# Kirby Calculus for Null-Homotopic and Null-Homologous Framed Links in 3-Manifolds

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## Abstract

In this thesis the relation between 3–manifolds and framed links is studied.

It is well known, that to every framed link $L$ in an arbitrary 3–manifold $M$ another 3–manifold $M_L$ can be associated by surgery. In 1978, Kirby proved a one-to-one correspondence between closed 3–manifolds and framed links in the three sphere up to the so called Kirby moves. A generalization to links in arbitrary closed 3–manifolds was given by Fenn and Rourke.

In this thesis we extend the result of Fenn and Rourke to 3–manifolds with boundary. More precisely, for two framed links $L$ and $L'$ in an arbitrary 3–manifold $M$ with non-empty boundary we give necessary and sufficient conditions on $M_L$ and $M_{L'}$ such that $L$ and $L'$ are related by Kirby moves.

We apply our theorem to framed links whose components are all null-homologous or more specifically null-homotopic. In addition, we introduce an IHX-move on null-homologous framed links in some specific 3–manifolds. This move is designed to kill one of the obstructions in our theorem. Our IHX-move is closely related to the 4–dimensional torus, the Jacobi identity and hence to the theory of finite type invariants.

The next main result is the development of refined Kirby calculus on the set of admissible framed links in 3–manifolds with free abelian first homology group. A link is called admissible, if it is null-homologous and its linking matrix coincides with the identity matrix up to sign. We prove that in this setting, up to some conditions, $M_L$ and $M_{L'}$ are diffeomorphic if and only if $L$ and $L'$ are related by stabilizations, band-slides, pair-moves, admissible IHX-moves and lantern-moves.

**Zusammenfassung**

Diese Dissertation untersucht den Zusammenhang zwischen 3-Mannigfaltigkeiten und Verschlingungen.

Jeder gerahmten Verschlingung $L$ in einer 3–Mannigfaltigkeit $M$ kann eine weitere 3–Mannigfaltigkeit $M_L$ zugeordnet werden. Kirby konnte 1978 beweisen, dass es zwischen geschlossenen 3–Mannigfaltigkeiten und gerahmten Verschlingungen in der drei Sphäre modulo den so genannten Kirby-Moves eine bijektive Abbildung gibt. Eine Verallgemeinerung für gerahmte Verschlingungen in beliebigen geschlossenen 3–Mannigfaltigkeiten zeigten Fenn und Rourke.

In dieser Dissertation erweitern wir das Resultat von Fenn und Rourke auf 3–Mannigfaltigkeiten mit Rand. Für zwei gerahmte Verschlingungen $L$ und $L'$ in einer 3–Mannigfaltigkeit mit Rand $M$ stellen wir notwendige und hinreichende Bedingungen an die 3–Mannigfaltigkeiten $M_L$ und $M_{L'}$ so, dass $L$ und $L'$ durch Kirby-Moves ineinander überführt werden können.

Unser Theorem wenden wir auf gerahmte Verschlingungen mit nur null-homologen Komponenten sowie auf den Spezialfall von nur null-homotopen Komponenten an. Zusätzlich definieren wir auf der Menge aller null-homologen gerahmten Verschlingungen den IHX-Move. Diese Bewegung ist so konstruiert, dass eine der gestellten Bedingungen in unserem Theorem überflüssig wird. Der IHX-Move resultiert aus der Henkelzerlegung des 4–dimensionalen Torus, ist verwandt mit der Jacobi-Identität und somit auch mit der Theorie der Invarianten von endlichem Typ.

Weiter entwickeln wir verfeinerte Bewegungen auf einer Teilmenge von ausgewählten null-homologen gerahmten Verschlingungen in 3–Mannigfaltigkeiten, deren erste Homologiegruppe eine freie abelsche Gruppe ist. Genauer gesagt untersuchen wir Verschlingungen, deren Verschlingungsmatrizen bis auf die Vorzeichen mit der Einheitsmatrix übereinstimmen. Für diesen Fall beweisen wir, dass zwei 3–Mannigfaltigkeiten $M_L$ und $M_{L'}$, bis auf gewisse Bedingungen, genau dann diffeomorph sind, wenn $L$ und $L'$ durch Stabilization, Band-Slides, Admissible-IHX-Moves und Lantern-Moves ineinander überführt werden können.

*To Beat and Valentina.*

# Contents

# Introduction

This thesis is devoted to the study of links in 3–manifolds. A link is an embedding of disjoint copies of the unit circle into a 3–manifold. One motivation to study knot theory is its remarkable connection to the theory of 3–manifolds.
In this work a 3–manifold is considered to be connected, compact and oriented.

Lickorish and Wallace independently showed that every closed 3–manifold can be obtained from a link in the 3–sphere $S^3$ by a process called surgery. In 1978, Kirby proved that there is a one-to-one correspondence between closed 3–manifolds and links in $S^3$ modulo the two Kirby moves called stabilization and handle-slides. Therefore, one can study closed 3–manifolds by analyzing link invariants stable under Kirby moves.

In 1984, Jones introduced a new polynomial invariant for links in $S^3$ called the Jones polynomial. This strong invariant had a huge impact on knot theory and was extended to an infinite family of quantum link invariants. To understand the structure of the set of quantum link invariants the notion of finite type invariants was introduced. Link invariants can be extended to immersed circles, i.e., links with double points. Then, an invariant is of type $m$ if it vanishes for links with $m + 1$ double points but not for links with $m$ double points. All quantum link invariants are of finite type. Moreover, they are unified in the Kontsevich integral which is proven to be the universal finite type invariant for links.

All 3–manifolds naturally organize into a category of 3–cobordisms by splitting their boundary into two parts: incoming and outgoing. Objects of the category of 3–cobordisms are surfaces, morphisms are given by 3–cobordisms (i.e., the 3–manifolds) and the composition is given by gluing along the boundary. In 1989, Witten initialized the study of topological quantum field theories (TQFTs) with Chern-Simons action. A TQFT is a monoidal functor from the category of cobordisms to the category of finite dimensional vector spaces. Witten's work inspired Reshetikin–Turaev to define an infinite family of 3–manifold invariants, called WRT-invariants. These invariants are given by linear sums of quantum invariants of the surgery link and they extend to TQFTs.

The next natural problem is to define a notion of "finite type" for 3–manifold invariants. The first step in this direction was made by Ohtsuki in 1996. Every integral homology sphere can be obtained from $S^3$ by surgery along an admissible framed (i.e., algebraically split and $\pm 1$–framed) link in $S^3$. Ohtsuki defined finite type invariants for integral homology spheres by considering admissible framed links in $S^3$. This approach was generalized by Cochran–Melvin for admissible framed links in any closed 3–manifold. Here, a link is called admissible if it is null-homologous, algebraically split and $\pm 1$–framed. Another filtration defined on the vector space spanned by all 3–manifolds with the same first homology group and linking form was independently established by Goussarov and Habiro at the end of the 20th century by introducing claspers.

Hence, to build a consistent theory of finite type 3–manifold invariants, surgery has to be considered not only in $S^3$ but in arbitrary 3–manifolds with possible non-empty boundary. Furthermore, the set of links should be restricted to admissible framed links. Handle-slides do not preserve admissibility in general. Thus, a refined Kirby calculus for admissible framed links is needed. In the case of integral homology spheres, Habiro established in 2006 a refined Kirby calculus for admissible framed links in $S^3$ by considering a move called band-slide. Habiro proved that two admissible framed links in $S^3$ are related by handle-slides and stabilization if and only if they are related by band-slides and stabilization.

A generalization of Kirby's Theorem to framed links in arbitrary closed 3–manifold was given by Fenn-Rourke. The case of framed links in 3–manifolds with boundary was given by Roberts. They considered framed links modulo the two Kirby moves and one additional move, called $K_3$–move. Fenn-Rourke also studied the equivalence relation on framed links in closed 3–manifolds generated only by stabilizations and handle-slides.

For a framed link $L$ in a 3–manifold $M$ let $M_L$ denote the result of surgery and let $W_L$ denote the 4–manifold obtained from $M \times I$ by attaching 2-handles on $M \times \{1\}$ along $L \times \{1\}$.

**Fenn-Rourke Theorem.** *Let $M$ be a closed 3–manifold, and let $L$ and $L'$ be two framed links in $M$. Then the following conditions are equivalent.*

*(i) $L$ and $L'$ are related by a sequence of handle-slides and stabilization,*

*(ii) there exist both an orientation-preserving homeomorphism $h\colon M_L \to M_{L'}$ and an isomorphism $f\colon \pi_1(W_L) \to \pi_1(W_{L'})$ such that the following diagram commutes*

$$\begin{tikzcd} \pi_1(M_L) \arrow[rr, "h_*"] \arrow[d] & & \pi_1(M_{L'}) \arrow[d] \\ \pi_1(W_L) \arrow[rr, "f"] & & \pi_1(W_{L'}) \\ & \pi_1(M) \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{0.1}$$

*and we have*

$$\rho_*([W]) = 0 \in H_4(\pi_1(W_L); \mathbb{Z}). \tag{0.2}$$

*Here, $W$ is the closed 4–manifold obtained by gluing $W_L$ and $W_{L'}$ along their boundary using $\mathrm{id}_M$ and $h$, and $[W] \in H_4(W; \mathbb{Z})$ is the fundamental class. The map $\rho_*$ is induced by a map $\rho\colon W \to K(\pi_1(W_L); 1)$ that is obtained by gluing natural maps from $W_L$ and $W_{L'}$ to $K(\pi_1(W_L); 1)$, where $K(\pi_1(W_L); 1)$ is an Eilenberg–Mac Lane space.*

Garoufalidis-Kricker showed that the Fenn-Rourke Theorem holds for 3-manifolds with connected boundary.

The main purpose of this thesis is to extend the Fenn-Rourke Theorem to 3–manifolds with arbitrary boundary and to refine it for admissible framed links.

Our first result is a generalization of the Fenn-Rourke Theorem to 3–manifolds with multiple boundary components. Then, we apply this result to null-homotopic framed links in 3–manifolds. In general, the obstruction $\rho_*([W]) = 0$ (0.2) is not easy to compute. We show that for null-homotopic links we have $H_4(\pi_1(W_L); \mathbb{Z}) = 0$ and therefore $\rho_*([W]) = 0$. Thus, our findings apply for example to surgery along null-homotopic framed links in cylinders over surfaces. Moreover, we refine Kirby calculus for null-homotopic admissible framed links in 3–manifolds with boundary.

Then, we extend these results to null-homologous framed links in 3–manifolds with boundary. Our main theorem is in between the two Fenn-Rourke statements. We allow a subset of the $K_3$–moves, i.e., $K_3$–moves that keep a link null-homologous, and one additional move, the IHX–move, that is designed to kill the obstruction $\rho_*([W]) = 0$. The IHX–move corresponds to an IHX–shaped clasper. Such IHX–shaped claspers stand in close relationship to the IHX–relation in the theory of finite type invariants which corresponds to the Jacobi identity, the key relation in the definition of a Lie algebra. Moreover, we show that the IHX–move is related to the handle decomposition of the 4–torus.

Our main theorem on the extension of Kirby calculus for null-homologous framed links is the following.

**Theorem 1.** *Let $M$ be a 3–manifold with $\partial M \neq \emptyset$ and $H_1(M;\mathbb{Z})$ free abelian. Let $P \subset \partial M$ be a subset containing exactly one point of each connected component of $\partial M$. Let $L$ and $L'$ be null-homologous framed links in $M$. Then the following conditions are equivalent.*

*(i) $L$ and $L'$ are related by a sequence of stabilization, handle-slides, null-homologous $K_3$–moves and IHX–moves.*

*(ii) there is an orientation-preserving homeomorphism $h\colon M_L \to M_{L'}$ restricting to the canonical identification $\partial M_L \cong \partial M_{L'}$ such that the following diagram commutes.*

$$\begin{tikzcd} H_1(M_L, P_L;\mathbb{Z}) \arrow[rr, "h_*"] \arrow[dr] & & H_1(M_{L'}, P_{L'};\mathbb{Z}) \arrow[dl] \\ & H_1(M, P;\mathbb{Z}) & \end{tikzcd} \qquad (0.3)$$

If the homology groups are considered with rational coefficients and $\mathbb{Q}$–null-homologous $K_3$–moves are allowed then a similar theorem holds for $\mathbb{Q}$–null-homologous links in arbitrary 3–manifolds.

Additionally, we give a refined Kirby calculus for admissible framed links in 3–manifolds with free abelian first homology group. Therefore, we modify the IHX–move and the null-homologous $K_3$–move to obtain moves on admissible links which we call *pair-move* and *admissible IHX–move*. Moreover, we need to allow a further move called *lantern-move*.

**Theorem 2.** *Let $M$ and $P$ be as in Theorem 1. Let $L$ and $L'$ be admissible framed links in $M$. Then the following conditions are equivalent.*

(i) *$L$ and $L'$ are related by a sequence of stabilization, band-slides, pair-moves, admissible IHX–moves and lantern-moves,*

(ii) *there is an orientation-preserving homeomorphism $h\colon M_L \to M_{L'}$ restricting to the canonical identification $\partial M_L \cong \partial M_{L'}$ such that Diagram 0.3 commutes.*

**Plan of the Thesis**

The thesis is divided into two parts. The first part provides background information on the techniques that are used. The second part consists of two individual articles presenting the results.

Part I is organized as follows. The first chapter presents knots and links and some basics of knot theory. The notion of surgery is introduced to establish the connection to 3–manifolds, and Kirby's main Theorem is stated.

In Chapter 2, the main definitions and statements of Kirby calculus are specified from the viewpoint of 4–manifold theory. Kirby diagrams and dotted circle diagrams are explained and all necessary formalisms of the Fenn-Rourke Theorem are provided.

Chapter 3 outlines Habiro's refined Kirby calculus for integral homology spheres. The main results are stated together with a short sketch of the proof. Variations of these statements and adaptions of the outlined proof will later be used to give a refined Kirby calculus for admissible framed links.

Chapter 4 introduces clasper calculus, a powerful tool to study 3–manifolds. After giving the essential definitions, the equivalence relations emerging from this calculus are compared to other equivalence relations. Moreover the notion of finite type invariant is introduced to emphasize the importance of the theory of clasper calculus. Finally, the definition of the IHX-move is given.

Part II splits into two chapters. Chapter 5 consists of the paper *"On Kirby calculus for null-homotopic framed links in 3–manifolds"*, a joint work with K. Habiro that will appear in the journal of Algebraic & Geometric Topology. This paper comprises the generalization of the Fenn-Rourke Theorem to manifolds with boundary as well as some applications to null-homotopic framed links.

Chapter 6 contains a preprint of the paper *"Kirby calculus for null-homologous framed links in 3–manifolds"*, a joint work with K. Habiro. In this paper we prove Theorem 1 and 2.

# Part I

# General Introduction

# Chapter 1

# Three manifolds and knots

This chapter introduces the basic concepts of knot theory and its connection to 3–manifold theory. We start by defining knots and links. Then, we show that any closed 3–manifold can be obtained by surgery along a framed link in $S^3$. Finally, we state Kirby's Theorem which gives a one to one correspondence between 3–manifolds and framed links in $S^3$ up to the so called Kirby moves.

## 1.1 Knots and links

Throughout Chapter 1 to Chapter 4 the following conventions are used. We are interested in topological 3–manifolds. If not otherwise stated a 3–manifold $M$ is always considered to be compact, connected and oriented. Two 3–manifolds $M$ and $M'$ are said to be equivalent $M \cong M'$ if there exists an orientation-preserving homeomorphism $h\colon M \to M'$.

For $n \in \mathbb{N}$, an *n–component link* in a 3–manifold $M$ is an embedding of $n$ disjoint copies of the unit circle $S^1$ into $M$, i.e., $L\colon S^1 \sqcup \cdots \sqcup S^1 \to M$. It is convenient to use the same symbol $L$ for the map and the image $L(S^1) \in M$. A one-component link is called a *knot*.
Two $n$-component links $L$ and $L'$ are said to be equivalent if there exists an ambient isotopy $F\colon M \times [0,1] \to M$ mapping $L$ to $L'$, i.e., a homotopy $F\colon M \times I \to M$ where each $F_t\colon M \to M$ is a homeomorphism with $F_0 = \mathrm{id}_M$ and $F_1(L) = L'$. We will not distinguish between a link and its equivalence class and denote both by $L$.

A link in the 3-sphere $S^3$ is called *polygonal* if its image in $S^3$ is the union of a finite set of line segments. Links which are equivalent to a polygonal link are called *tame*, otherwise they are called *wild*. Wild knots or links are not studied in this thesis. From now on, a link in $S^3$ is always considered to be tame.

For a link in $S^3$ we can consider a projection to the plane $\mathbb{R}^2$. A projection is called *regular* if no three points of the polygonal link are mapped to the same point and no vertex is mapped onto a double point. At each double point of a projection the distances between the two intersecting link segments and the plane can be determined. We can encode this information into the projection by creating a break in the segment that is closer. A regular projection of a link with this information at each double point, i.e., which segment is over and which is under crossing, is called a *link diagram*. A double point in a link diagram is called a *crossing*. A link diagram determines

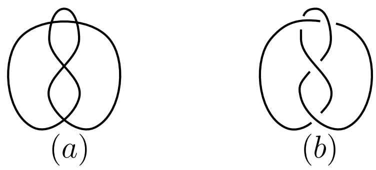



Figure 1.1: (a) A regular projection. (b) A knot diagram.

a link up to its equivalence class. Therefore, knots and links in $S^3$ can be studied by their diagrams.
There are three moves defined on link diagrams, called *Reidemeister moves* RI, RII and RIII, see Figure 1.2. These three moves do not change the equiv-

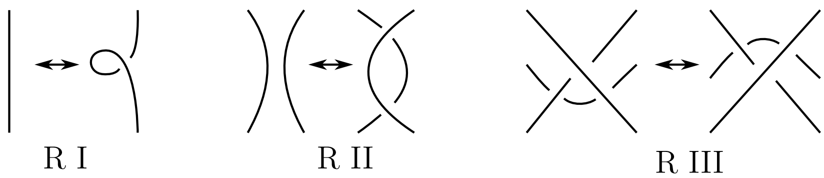



Figure 1.2: The three Reidemeister moves R I, R II and R III.

alence class of the link represented by the diagram. Moreover, two links $L$ and $L'$ in $S^3$ are equivalent if and only if their link diagrams are related by a sequence of Reidemeister moves.

To each crossing in a diagram of an oriented link $L$ in $S^3$ a sign $\pm 1$ is assigned according to the convention shown in Figure 1.3. The *linking number* $\mathrm{lk}(L_i, L_j)$ of two distinct link components $L_i$ and $L_j$ is half the sum of all the signs of those crossings where both $L_i$ and $L_j$ are involved. One can check that the linking number does not change under R I, R II and RIII and is therefore an invariant of the link.

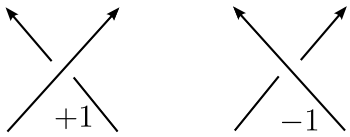



Figure 1.3: Sign convention for oriented link diagrams.

## 1.2 Surgery description

Studying knots and links is of special interest since there is a one-to-one correspondence between 3-manifolds and links up to Kirby moves. Let us give some necessary definitions.

For $n \in \mathbb{N}$, let $L = L_1 \sqcup \cdots \sqcup L_n$ be a link in a 3–manifold $M$. For each component $L_i$ denote by $\partial N(L_i)$ the boundary of a tubular neighborhood $N(L_i)$ of $L_i \in M$. The link $L$ is called a *framed link* if to each component $L_i$ an essential, i.e., non-separating, simple closed curve $f_i \subset \partial N(L_i)$ is assigned. The *framing curve* $f_i \subset \partial N(L_i)$ is considered up to homotopy.
A new manifold $M_L$ can be obtained from $M$ by *Dehn surgery* along $L$ as follows. For $i = 1, \ldots n$, remove the interior of $N(L_i)$ in $M$ and glue in a solid torus $D^2 \times S^1$ along the boundary $\partial N(L_i)$ such that the meridian $\partial D^2 \times \{*\}$, $* \in S^1$, is mapped to the framing curve $f_i \in \partial N(L_i)$. This type of surgery is also called *rational surgery.*

For a knot $K$ in $S^3$ we define the *knot exterior* $X$ as the closure of $S^3 \setminus N(K)$. In this case, a canonical *longitude* $l$ of the boundary $\partial X$ can be defined as a curve homologically trivial in $X$. Such a canonical longitude is unique up to isotopy. The *meridian* $m$ of the knot $K$ is a curve in $\partial X$ representing the generator of $H_1(X)$. Up to isotopy, any essential simple closed curve $c \subset \partial X \cong \partial N(K)$ can be uniquely described in therms of the meridian $m$ and the longitude $l$ by $c = p \cdot m + q \cdot l$ for some coprime integers $p, q \in \mathbb{Z}$. Therefore, we can describe the framing curve by a reduced fraction $\frac{p}{q}$, where we set $\frac{1}{0} = \infty$. The manifolds $S^3_K$ is said to be obtained from $S^3$ by rational surgery along the $\frac{p}{q}$-framed knot $K \subset S^3$. If $q = 1$, the surgery is called *integral.* The definition of rational surgery can be naturally extended to links in $S^3$.

The following theorem shows the importance of surgery.

**Theorem 3** (Lickorish, Wallace)**.** *Every closed 3–manifold $M$ can be obtained from $S^3$ by integral surgery along a framed link $L$ in $S^3$, $M \cong S^3_L$.*

*Proof.* See Lickorish [22] or Wallace [35], or for a comprehensive proof consult the book of Saveliev [33, Theorem 2.1]. □

Let $M$ be a closed 3–manifold obtained by Dehn surgery along an $n$–component framed link $L = L_1 \sqcup \cdots \sqcup L_n$ in $S^3$, i.e., $M = S^3_L$. We diagramatically describe $M$ by the link diagram of $L$ where each link component $L_i$ is labelled by its *framing coefficient* $\frac{p_i}{q_i}$ for $i = 1, \ldots, n$.

**Example 4.** Rational surgery along the $\frac{p}{q}$-framed unknot in $S^3$ gives the lens space $L(p, q)$. For the $\pm 1$-framed unknot $U$ we get $S^3_U \cong S^3$.

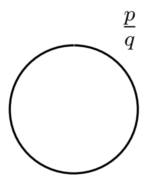



Figure 1.4: The lens space $L(p, q)$.

**Example 5.** Surgery along the $+1$-framed right-handed trefoil in $S^3$ gives the Poincaré manifold, see Figure 1.5.

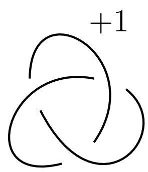



Figure 1.5: The Poincaré manifold by a diagram with framing coefficients

Let $L$ be a link in $S^3$. In a link diagram of $L$, we can consider for each link component $L_i$ a curve $f_i$ parallel to $L_i$. The parallel curves define framing curves $f_i$ in $S^3$ for each component $L_i \in S^3$. The induced integral framing is called *blackboard framing.* If no number is assigned to a diagram of a framed link, the blackboard framing is assumed. Using the blackboard framing the framing coefficient then equals the linking number $\mathrm{lk}(L_i, f_i)$ where an orientation of $L_i$ induces an orientation on the curve $f_i$. An example illustrating the blackboard framing is given in Figure 1.6.

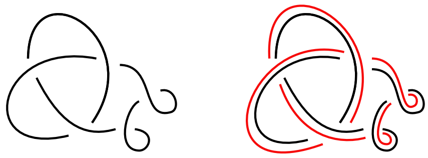



Figure 1.6: The Poincaré manifold by using a diagram with blackboard framing. The red curve in (b) is the framing curve corresponding to the blackboard framing.

The blackboard framing depends on the diagram and changes under a Reidemeister RI move. Consider the move RI' shown in Figure 1.7 . Then, two diagrams with blackboard framing represent the same framed link in $S^3$ if and only if they are related by RI', RII, and RIII moves.

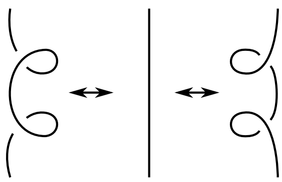



Figure 1.7: The Reidemeister RI' move for framed links.

Henceforth, we only consider framing curves $f_i$ which are homotopic to $L_i$ in $\partial N(L_i)$. By *surgery along a framed link* $L$ in a 3-manifold $M$ we mean Dehn surgery along $L$ with framing curves $f_i$ homotopic to $L_i$ in $N(L_i)$. For links in $S^3$ this corresponds to integral surgery. We can, by this convention, omit the therm “integral”.

### 1.2.1 Kirby theorem

Knowing that any 3–manifold $M$ can be described as $M = S^3_L$ for some framed link $L \subset S^3$, the natural question to ask is when do two framed links

in $S^3$ represent homeomorphic manifolds.
In 1978, R. Kirby proved a one-to-one correspondence between 3–manifolds and framed links in $S^3$ up to two moves, known as the *Kirby moves*. The two Kirby moves are:

- *stabilization* (**K1**): adding or removing an isolated $\pm 1$-framed unknot.
- *handle-slide* (**K2**): sliding a component $L_i$ over a component $L_j$, i.e., the new component is $L_i' = L_i \#_b L_j$ where $b$ is a band connecting the two components, see Figure 1.8. All other components of the link $L$ are unchanged.

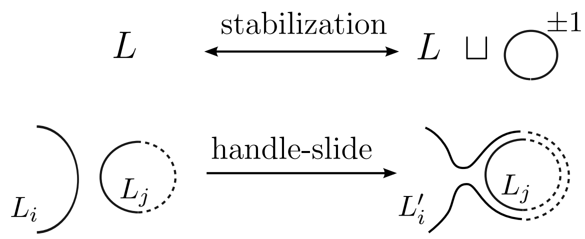



Figure 1.8: Stabilization and handle-slides.

If the framings of $L_i, L_j \subset S^3$ are given by the integers $f_i, f_j \in \mathbb{N}$ then the framing of $L_i'$ can be computed as:

$$f_i' = f_i + f_j + 2lk(L_i, L_j).$$

To to compute $lk(L_i, L_j)$ an orientation on $L_i$ and $L_j$ is chosen such that the connected band sum induces an orientation on $L_i'$.

**Theorem 6** (Kirby Theorem)**.** *Two framed links $L$ and $L'$ in $S^3$ have orientation-preserving homeomorphic results of surgery if and only if they are related by a sequence of stabilization and handle-slides.*

*Proof.* See the original paper of Kirby [17]. In this thesis we follow the proof given by Fenn–Rourke [7]. The main idea is to consider 3–manifolds as the boundary of some 4–manifolds and apply Cerf theory. More details are given in Chapter 2.
An alternative proof based on the stable equivalence of Heegard splittings and Wajnryb's presentation for the mapping class group of a surface is given in [25, 23]. □

Kirby's Theorem shows that 3–manifolds can be studied by analyzing link invariants that are stable under Kirby moves. Unfortunately the Kirby moves do not preserve the classical link invariants. It is hard to find link invariants that are stable under Kirby moves. In fact, there are infinitely many framed links representing the same 3–manifold.

**Example 7.** The three framed links shown in Figure 1.9 in $S^3$ can all be related to the empty knot by a sequence of handle-slides and stabilization. See [33, Section 3.2] for a detailed analysis. Therefore, the result of surgery along these links is always $S^3$.

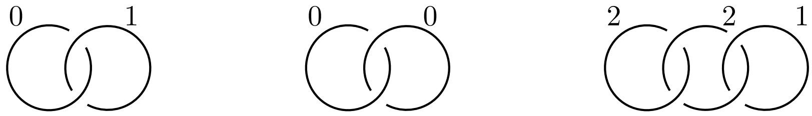



Figure 1.9: Three framed links, each representing $S^3$.

Fenn and Rourke showed in [7] that the two Kirby moves are equivalent to a move called *Fenn–Rourke–move*, see Figure 1.10.

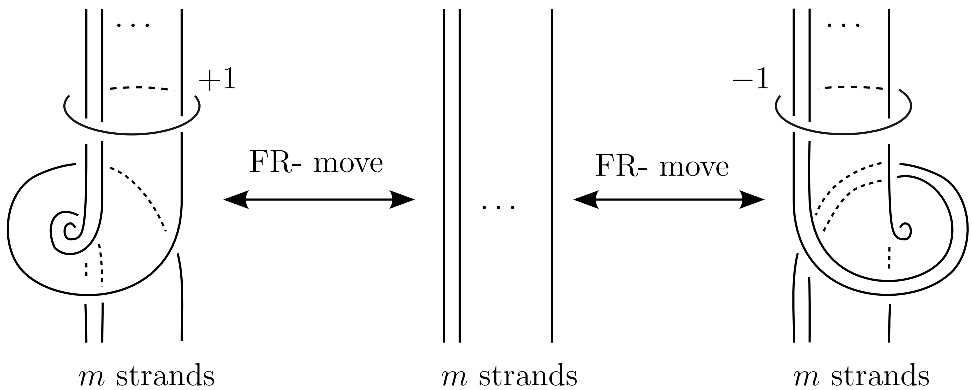



Figure 1.10: The Fenn–Rourke–move

# Chapter 2

# Kirby calculus and 4–manifolds

In this chapter we introduce handle decompositions for $n$–manifolds. Then, we characterize 4–manifolds by certain link diagrams called Kirby diagrams. Moreover, all the necessary notations are given to precisely state the Fenn–Rourke Theorem, a generalization of Kirby's Theorem for links in 3–manifolds other than $S^3$.

## 2.1 Handle decomposition

Let $M$ be an $n$–dimensional manifold. For $0 \leq k \leq n$, we define a $k$-handle $h_k$ *of dimension* $n$ as a copy of $D^k \times D^{n-k}$ attached to the boundary of $M$ along $\partial D^k \times D^{n-k}$ by an embedding $\varphi \colon \partial D^k \times D^{n-k} \to \partial M$. We look at the union $M \cup_\varphi h_k$ as an $n$–manifold. The embedding $\varphi$ is determined by an embedding $\varphi_0 \colon S^{k-1} \hookrightarrow \partial M$ with a given framing, i.e., a homotopy class of trivializations of the normal bundle of $\mathrm{Im}(\varphi_0)$.
The $k$–disk $D^k \times \{0\}$ is called *core* of the handle $h_k$ and $\{0\} \times D^{n-k}$ is called *cocore*. The embedding $\varphi$ is called *attaching map*, $\partial D^k \times D^{n-k}$ the *attaching region*, $\partial D^k \times \{0\}$ the *attaching sphere* and $\{0\} \times \partial D^{n-k}$ the *belt sphere*.

**Example 8.** In dimension 2 we have the following handles:

- A 0-handle $h_0$ is a disk $\{pt\} \times D^2$ with attaching region the empty set.
- A 1-handle $h_1$ is a square $D^1 \times D^1 \cong [-1, 1] \times [-1, 1]$ with attaching region two opposite sides of the square $\{-1\} \times [-1, 1]$ and $\{1\} \times [-1, 1]$.
- A 2-handle $h_2$ is a disk $D^2 \times \{pt\}$ with attaching region $S^1$.

Figure 2.1 illustrates the anatomy of a 2-dimensional 1-handle.

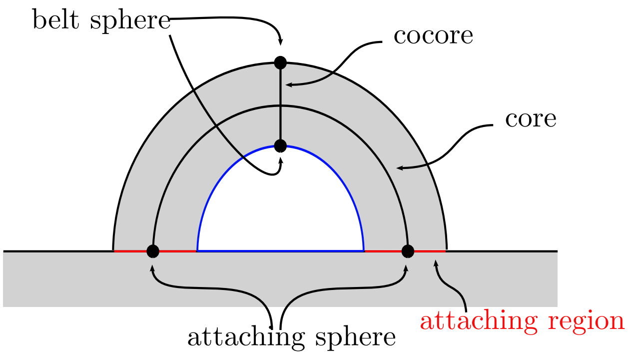



Figure 2.1: A 2–dimensional 1-handle.

A manifold $M$ with boundary $\partial M = M_- \sqcup M_+$ (the disjoint union of two compact submanifolds) is called a *handlebody relative to* $M_-$ if it can be constructed by attaching handles to $M_- \times I$, where $I = [0, 1]$. This construction is called a *handle decomposition*. If $M_- = \emptyset$, then $M$ is called a *handlebody*.

Remark: This generalizes the notion of a 3–dimensional handlebody $V_g$ that is obtained from $D^3$ by attaching $g$ 1-handles. Note that the boundary $\partial V_g$ is a closed surface of genus $g$.

**Proposition 9.** *Every smooth compact manifold $M$ with boundary $\partial M = M_- \sqcup M_+$ as above admits a handle decomposition as follows:*

$$M_0 \subset M_1 \subset M_2 \cdots \subset M_n = M,$$

*where $M_0 = M_- \times I$ and $M_k$ is obtained from $M_{k-1}$ by attaching $k$-handles.*

*Proof.* The proof of this result relies on Morse theory. The standard references are [27, 28]. □

One can relate Morse theory to handle decompositions roughly as follows. Let $M$ be a smooth manifold. A smooth function $f\colon M \to \mathbb{R}$ is called a *Morse function* if all its critical points are non-degenerate, i.e., the Hessian matrix is non-degenerate at all critical points. Let $M$ be a relative handlebody $M$ with $\partial M = M_- \sqcup M_+$. By Morse theory, every smooth real valued function $f$ on $M$ with $f^{-1}(0) = M_-$ and $f^{-1}(1) = M_+$ can be approximated by a Morse function $\tilde{f}$. Then, every critical point of the Morse function $\tilde{f}$ of index $k$ corresponds to a $k$-handle.

**Example 10.** We illustrate a handle decomposition of the 2–dimensional torus $T^2$ in Figure 2.2. First, a disk is added to the empty set, then two 1-handles are attached and finally another disk is attached along the boundary circle to obtain the closed 2–manifold $T^2$.

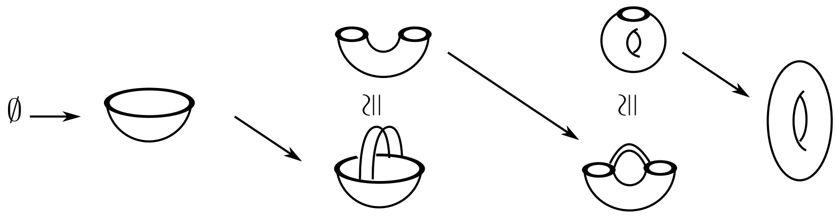



Figure 2.2: Handle decomposition of $T^2$.

Note that a handle decomposition is not unique. To begin with, it is always possible to construct a so called *cancelling handle pair*.

**Proposition 11.** *If the attaching sphere of a $k$-handle $h_k$ intersects the belt sphere of a $(k-1)$-handle $h_{k-1}$ transversely in a single point then $h_k$ and $h_{k-1}$ form a cancelling pair of handles. Thus we can omit $h_k$ and $h_{k-1}$ in the handle decomposition.*

*Proof.* For a sketch see [11, Proposition 4.2.9], or [28, Theorem 5.4] for a careful proof. □

**Example 12.** If in Figure 2.1 a 2-handle (which is a disk) is attached along the blue curve, the obtained manifold would be homeomorphic to the original one. The 1-handle and the disk attached in this way form a cancelling handle pair.

There is a second operation that changes the handle decomposition but not the handlebody. Consider two $k$-handles $h_k$ and $h'_k$. We can slide $h_k$ over $h'_k$ by pushing the attaching sphere of $h_k$ through the belt sphere of $h'_k$, see Figure 2.3. This operation is called a *handle-slide*. We will show in the next section how this handle-slide is related to the $K2$–move introduced in Chapter 1.

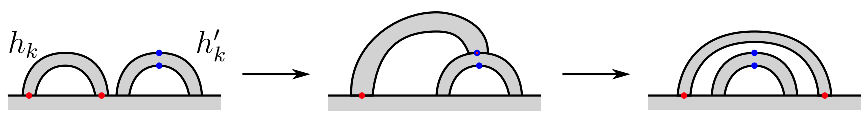



Figure 2.3: Illustration of a handle-slide in the case of 1-handles of dimension 2.

**Theorem 13** (Cerf)**.** *Two handle decompositions of a manifold $M$ are related by a sequence of handle-slides, creating/annihilating cancelling handle pairs and isotopies.*

*Proof.* See [11, Proposition 4.2.12] for a sketch, or Cerf's original proof [2]. □

## 2.2 Kirby diagrams for 4–manifolds

In this section, we study handle decompositions of 4–manifolds.

A handle decomposition of a smooth connected 4–manifold $W$ with one connected boundary component can be described as follows. Since $W$ is connected, we can assume that there is a unique 0-handle $h_0 = D^4$. Its boundary is $S^3 = \mathbb{R}^3 \cup \{\infty\}$. Thus, we draw the attaching spheres of the remaining handles in $\mathbb{R}^3$. Each 1-handle $h_1 = [0,1] \times D^3$ of $W$ is attached to $h_0 = D^4$ along two disjoint 3-balls $(\{0\} \times D^3) \sqcup (\{1\} \times D^3)$. Roughly speaking, attaching a 1-handle along two 3–balls is equivalent to identifying the two 3–balls with each other. Thus, we draw the 3–balls $(\{0\} \times D^3) \sqcup (\{1\} \times D^3)$ and we can assume that the identification is given by the reflection through the plane that perpendicularly bisects the segment joining the centers of the 3-balls, see Figure 2.4

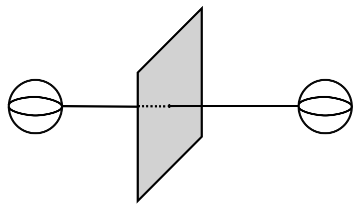


Figure 2.4: 1-handles: Identification of the attaching spheres.

Let $W_1$ be a manifold obtained by attaching 1-handles to $D^4$. The embedding of a 2-handle is given by the image $\varphi\colon S^1 \hookrightarrow \partial W_1$ and a framing curve. Thus, any framed knot $K$ in $\partial W_1$ whose framing curve is homotopic to $K$ in $N(K)$ defines an embedding of a 2-handle. Note that, a knot in $\partial W_1$ can intersect a 1-handle. Suppose the knot intersects the attaching region $D^3$ of the 1-handle in $S^3$. Since we think of the two attaching regions $D^3 \sqcup D^3 \in S^3$ to be identified, the knot goes once through the 1-handle and emerges at the second attaching sphere. This can best be seen by an example. Figure 2.5 shows an example of the attaching spheres of two 1-handles (the top two balls form a pair and the two balls on the bottom another one) and three 2-handles. A diagram showing the attaching regions of the 1-handles and the attaching spheres of the 2-handles is called a *Kirby diagram*.

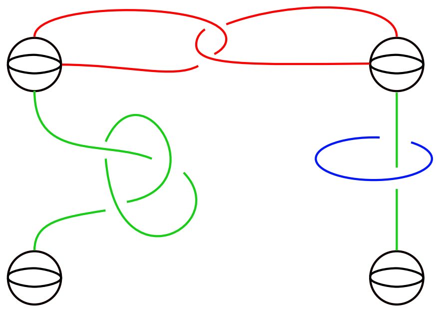


Figure 2.5: A Kirby diagram with two 1-handles and three 2-handles.

In this notation, there is no canonical way to assign framing coefficients for 2-handles running through a 1-handle. To see this, consider Figure 2.6 where a 2-handle goes once through a 1-handle and the framing is given by the red curve. After moving the attaching sphere of the 2-handle, i.e., the knot around the attaching sphere of the 1-handle as depicted, the linking number of the framing curve and the knot has changed.

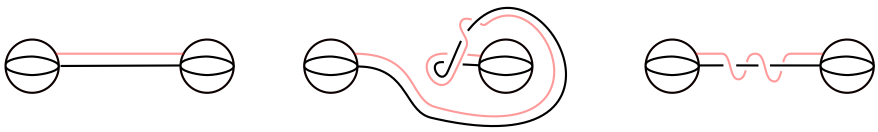


Figure 2.6: The linking number of a parallel curve can be changed by an isotopy. Here, the red curve represents the framing curve.

**Remark.** The Kirby diagram shown in Figure 2.6 is a cancelling handle pair. The attaching sphere of the 2-handle, i.e., the knot, goes exactly once through the 1-handle and therefore intersects the belt sphere of the 1-handle transversely in one point. By Proposition 11 this is a cancelling handle pair.

For a closed 4–manifold $W$ with handle decomposition

$$D^4 = M_0 \subset M_1 \subset M_2 \subset M_3 \subset M_4 = W,$$

the boundary $\partial M_2$ is homeomorphic to $\partial M_1$ and there is a canonical way to attach the 3– and 4-handles. The 3– and 4-handles can be considered as duals of the 1– and 0-handles. A proof is given in [19]. In terms of Morse theory, this can be seen by replacing the Morse function $f$ by $1 - f$. Thus, a smooth closed 4–manifold is completely determined by its 1– and 2-handles, i.e., a Kirby diagram as in Figure 2.5 determines a closed 4–manifold.

**Example 14.** (a) Attaching a 2-handle along a 0–framed unknot in $S^3 = \partial D^4$ gives $S^2 \times D^2$.

(b) The closed 4–manifold described by the $\pm 1$–framed unknot is $\pm\mathbb{C}P^2$, i.e., the complex projective plane or the complex projective plane with reversed orientation.

These examples are explained nicely in [33].

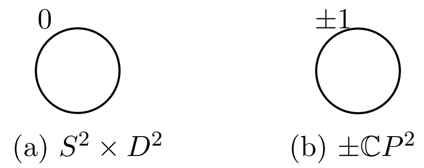



Figure 2.7: The 4–manifold $S^2 \times D^2$, and $\pm\mathbb{C}P^2$.

**Example 15.** The 4–torus $T^4 = S^1 \times S^1 \times S^1 \times S^1$ plays an important role in the computations of the refined Kirby calculus for null-homologous links. Figure 2.8 shows a standard Kirby diagram of $T^4$. This handle decomposition of $T^4$ consists of one 0-handle, four 1-handles, six 2-handles and by duality four 3-handles and one 4-handle. In this example the framing of the 6-component link $L = L_1 \sqcup \cdots \sqcup L_6$ is given as follows. Note that each component $L_i$ is null-homotopic and therefore bounds a surface $S_i$ in $D^4 \cup$ (1-handles). Then, the framing curve $f_i$ of the component $L_i$ is given by $f_i = \partial N(L_i) \cap S_i$. A detailed explanation is given in [1, Section 4.1].

### 2.2.1 Relating 3– and 4–manifolds

Now, we relate 3– and 4–manifolds by using Kirby diagrams.

The boundary of a 4–manifold $\mathsf{W}$ is a disjoint union of 3–manifolds. Let us now assume that $\partial\mathsf{W} = M$ is connected. Consider a framed link $L$ in $M$. If we attach 2-handles along $L$, we obtain a new 4–manifold $\mathsf{W}_L$. Roughly speaking, the boundary changes as follows. Attaching a 2-handle $h_2 = D^2 \times D^2$ along a component $L_i$ covers $N(L_i) = S^1 \times D^2$ that is therefore removed from $\partial\mathsf{W} = M$ while a new boundary part $D^2 \times S^1$ is attached to $M \setminus N(L_i)$. Therefore, the boundary of $\mathsf{W}_L$ is the same as the result of surgery along $L$ in $M$, i.e., $\partial\mathsf{W}_L = M_L$ where the notations are as introduced in Chapter 1.

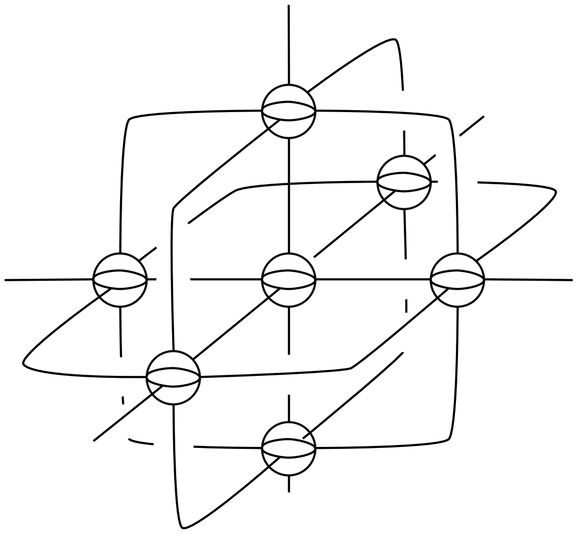


Figure 2.8: Standard 4-torus Kirby diagram.

It is natural to ask which 3–manifolds bound a 4–manifold.

By Theorem 3 every closed 3–manifold can be obtained by integral surgery along a link in $S^3$. Thus, we have the following corollary.

**Corollary 16.** *Every closed* 3*–manifold bounds a compact, oriented* 4*–manifold.*

*Proof.* This follows from Theorem 3. There also exists a direct proof due to Rohlin [32] by means of 4–manifold techniques. □

Let us quickly restate this result in the language of cobordism theory. Let $M$ and $M'$ be two closed 3–manifolds. An oriented compact smooth 4–manifold $W$ is called a *cobordism* between $M$ and $M'$ if $\partial W = (-M) \sqcup M'$, where $-M$ is $M$ with reversed orientation. Two closed 3–manifolds $M$ and $M'$ are called *cobordant*, if there exists a cobordism between them. If $W$ has just one boundary component $M$, then $M$ is cobordant to the empty set and we say, $M$ is *cobordant to zero*. Thus, by the previous corollary any closed 3–manifold is cobordant to zero.

## 2.3 Dotted circle notation

The manifold $M$ that is obtained by attaching $m$ 1-handles to $D^4$ is homeomorphic to $\#^m D^3 \times S^1$, the connected sum of $m$ copies of $D^3 \times S^1$. Here, $M$ can also be obtained from $D^4 \cong D^3 \times D^1$ by removing $m$ properly embedded disks $D^2$ from $D^4$. It is enough to specify the boundary $S^1$ of the disk $D^2$ that is carved out. Thus we can draw an unknot for each 1-handle and put a dot on it to distinguish it from the attaching spheres of the 2-handles. We will call a diagram with this notation *dotted circle diagram*. Observe that any path going through a dotted circle is going once through the associated 1-handle. An advantage of dotted circle diagrams is that framing coefficients are well defined and we can use the blackboard framing.

To transform a usual Kirby diagram into a dotted circle diagram, we draw a reference arc for each 3-ball pair to indicate how they are joined together. By isotopy, we can move the balls along the reference arc until they are next to each other. Then we switch to the dotted circle notation.

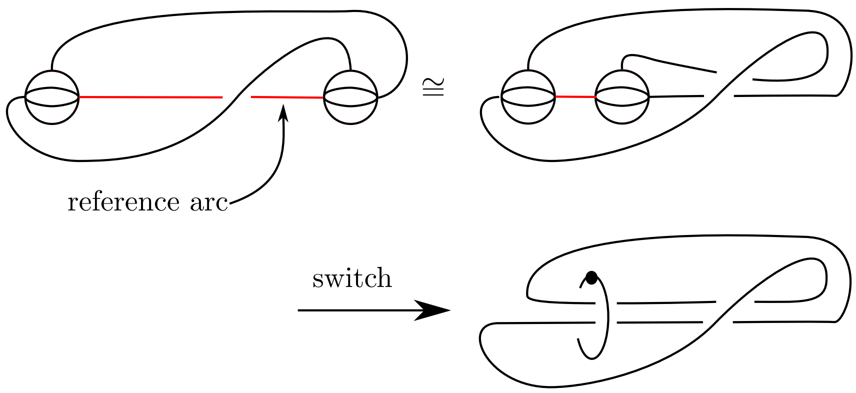
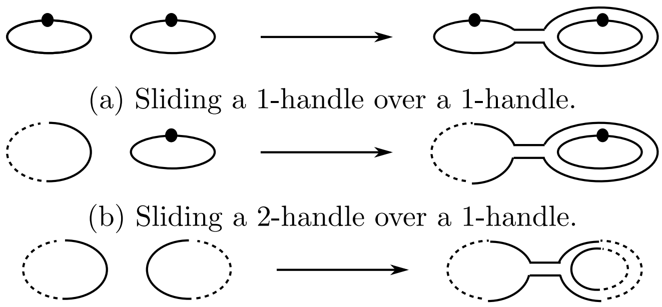



Figure 2.9: From a Kirby diagram to the dotted circle notation.

There is a choice for the reference arc. But the 4–manifold described is independent of the chosen reference arc. For dotted circle diagrams there exist sliding rules shown in Figure 2.10. By switching between Kirby diagrams and dotted circle notation one can show that the sliding rules do not change the 4–manifold, see [11, Section 5.4] or [1, Section 1.2] for more explanations.

S. Akbulut shows in [1, Chapter 4] how to obtain a dotted circle diagram for the 4-torus starting with the standard $T^4$ Kirby diagram. As the Kirby diagram of $T^4$ already consists of a 6–component link the intermediate steps get quite entangled. It is helpful, to simplify each step by isotopy. For the identification of the 1-handles we can flatten the balls as in Figure 2.11, The boxes can then be joined together along the reference arc by introducing the dotted circle. After carefully isotoping the link, Figure 2.12 is obtained.



(a) Sliding a 1-handle over a 1-handle.

(b) Sliding a 2-handle over a 1-handle.

(c) Sliding a 2-handle over a 2-handle.

Figure 2.10: Sliding rules for 1– and 2-handles.

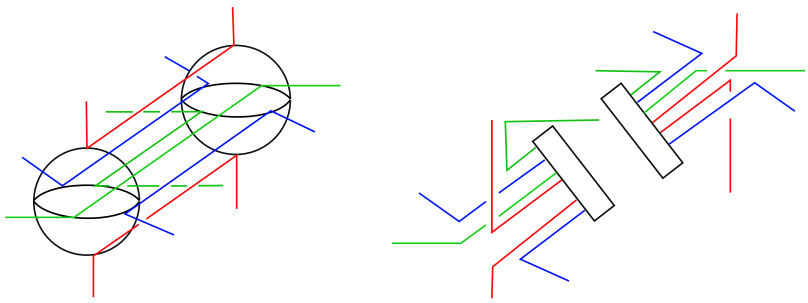


Figure 2.11: Flatten the 3–balls.

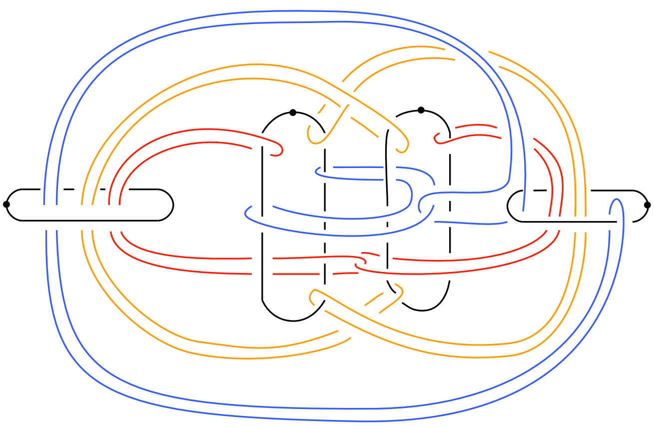


Figure 2.12: A dotted circle diagram for $T^4$.

This presentation of $T^4$ is closely related to the IHX-move that will be introduced in Chapter 4.

## 2.4 Surgery on links in closed 3–manifolds

Kirby's Theorem holds for links in $S^3$. For links in arbitrary closed 3–manifolds there exists a generalization due to Fenn and Rourke [7] which we state in this section.

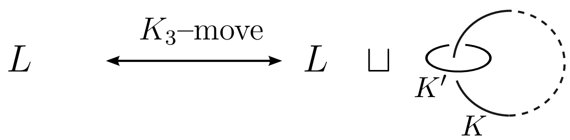
### 2.4.1 The $K_3$–move

Fenn and Rourke introduced a new move the $K_3$-move which we introduce now.

Recall that $D^4 \cup$ (1–handles) $\cong \#^m D^3 \times S^1$, where $m$ is the number of 1-handles attached. This manifold has the same boundary as $\#^m S^2 \times D^2$, which is obtained by attaching 2-handles along an $m$–component, 0–framed unlink. Therefore, if we are only interested in the 3–manifold, i.e., the boundary, we can replace the 1-handles by 0–framed unknots. For a Kirby diagram, with 1– and 2-handles, we consider the dotted circle notation. The boundary of the manifold obtained by attaching 1– and 2-handles is the same as the boundary of the manifold obtained by attaching 2-handles to the same diagram with all dotted circles replaced by 0–framed unknots.

A third Kirby move is defined as:

- $K_3$*–move*[1]: inserting/ deleting a pair $(K, K')$, where $K$ is a knot of arbitrary framing, and $K'$ is the 0–framed meridian of a tubular neighborhood $N(K)$ of $K$.




**Proposition 17.** *Let $L$ be a link in a 3–manifold $M$. Then, a $K_3$–move on $L$ does not change the result of surgery.*

[1] This move is also often called *circumcision move*.

*Proof.* We sketch how to show $M_L \cong M_{L \sqcup K \sqcup K'}$. Consider the 3–manifold $M$ as the boundary of a 4–manifold $\mathsf{W}$. Then $M_{L \sqcup K \sqcup K'} \cong \partial(\mathsf{W}_{L \sqcup K \sqcup K'})$, where $\mathsf{W}_{L \sqcup K \sqcup K'}$ is obtained from $\mathsf{W}$ by attaching 2-handles along $L \sqcup K \sqcup K'$. The boundary is not changed if the 0–framed unknot $K'$ is replaced by a dotted circle $\widetilde{K'}$. Here $\widetilde{K'}$ is a 1-handle and thus

$$\partial(\mathsf{W}_{L \sqcup K \sqcup K'}) \cong \partial(\mathsf{W}_{L \sqcup K \sqcup \widetilde{K'}}).$$

The pair $(K, \widetilde{K'})$ is a cancelling handle pair and therefore $\mathsf{W}_L \cong \mathsf{W}_{L \sqcup K \sqcup \widetilde{K'}}$. To summarize, we have

$$M_{L \sqcup K \sqcup K'} \cong \partial(\mathsf{W}_{L \sqcup K \sqcup K'}) \cong \partial(\mathsf{W}_{L \sqcup K \sqcup \widetilde{K'}}) \cong \partial(\mathsf{W}_L) \cong M_L.$$

See also [7, Proof of Theorem 8] □

**Remark.** The $K_3$–move is redundant in $S^3$. Consider a blackboard framed link diagram of a pair $(K, K') \subset S^3$. A handle-slide of $K$ over the 0–framed meridian $K'$ changes a crossing of $K$, see Figure 2.13. Thus, we can unknot the component $K$ and get a Hopf-link with framing $(0, 0)$ or $(1, 0)$. This framed Hopf-link is related to the empty link by a sequence of handle-slides and stabilization by Example 1.9.

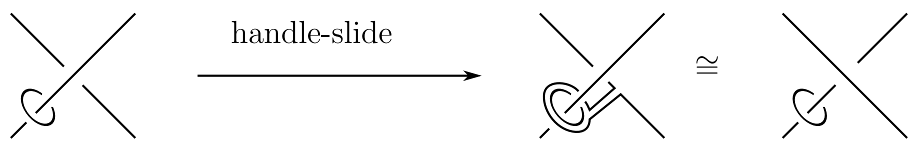



Figure 2.13: Unknotting process by handle-sliding over the 0–framed meridian.

## 2.4.2 The Fenn–Rourke Theorem

In this section we consider the relation on framed links in an arbitrary closed 3–manifold given by stabilizations and handle-slides. First we introduce some notation and then we state the Fenn–Rourke Theorem.

Let $M$ be a closed 3–manifold and let $L$ be a framed link in $M$. We denote by $W_L$ the 4–manifold obtained from $M \times I$ by attaching 2-handles along $L \times \{1\} \subset \partial(M \times I)$. Note that $\partial W_L \cong M \sqcup M_L$. Thus $W_L$ is a cobordism between $M$ and $M_L$.

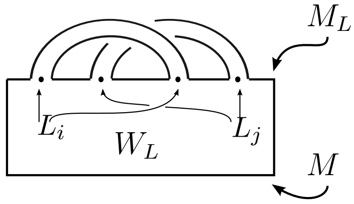



Figure 2.14: The cobordism $W_L$.

The inclusions $M_L \hookrightarrow W_L \hookleftarrow M$ induce surjective homomorphisms on the level of fundamental groups

$$\pi_1(M_L) \twoheadrightarrow \pi_1(W_L) \twoheadleftarrow \pi_1(M).$$

The kernel of the homomorphism $\pi_1(M) \to \pi_1(W_L)$ is generated by the homotopy classes of components of $L$. An Eilenberg–Mac Lane space $K(\pi_1(W_L), 1)$ can be obtained from $W_L$ by attaching cells which kill higher homotopy groups. Thus, there is a natural inclusion

$$\rho_L \colon W_L \hookrightarrow K(\pi_1(W_L), 1).$$

Now, consider two framed links $L$ and $L'$ in a closed, oriented 3–manifold $M$, and suppose that there exists a homeomorphism $h \colon M_L \to M_{L'}$. Moreover, we assume that there exists an isomorphism $f \colon \pi_1(W_L) \to \pi_1(W_{L'})$ such that the diagram

$$\begin{tikzcd} \pi_1(M_L) \arrow[rr, "h_*"] \arrow[d] & & \pi_1(M_{L'}) \arrow[d] \\ \pi_1(W_L) \arrow[rr, "f"] & & \pi_1(W_{L'}) \\ & \pi_1(M) \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{2.1}$$

commutes.

Define a 4–manifold $W$ by

$$W := W_L \cup_\partial (-W_{L'}),$$

where we glue $W_L$ and $-W_{L'}$ (the orientation reversal of $W_{L'}$) along the boundaries using the identity map on $M$ and the homeomorphism $h \colon M_L \xrightarrow{\cong} M_{L'}$, see Figure 2.15.

If we have a commutative Diagram (2.1), we can construct a map

$$\rho \colon W \to K(\pi_1(W_L), 1),$$

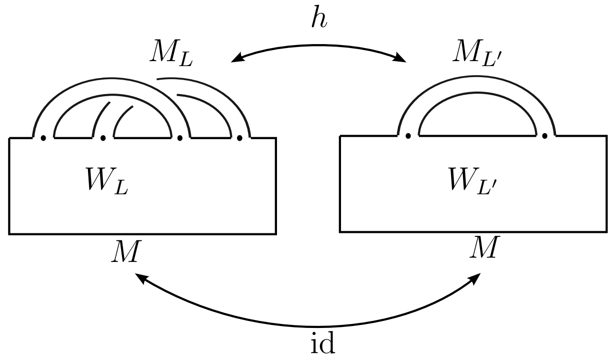



Figure 2.15: The closed 4–manifold $W := W_L \cup_\partial (-W_{L'})$.

by identifying $K(\pi_1(W_L), 1)$ and $K(\pi_1(W_{L'}), 1)$ (up to homotopy) using $f$ and combining the two maps $\rho_L$ and $\rho_{L'}$ as follows

$$\rho = \begin{cases} \rho_L & \text{on } W_L \\ f^{-1} \circ \rho_{L'} & \text{on } W_{L'}. \end{cases}$$

Let

$$\rho_* \colon H_4(W; \mathbb{Z}) \to H_4(\pi_1(W_L); \mathbb{Z})$$

be the induced map on homology. The fundamental class $[W] \in H_4(W)$ is mapped to a homology class

$$\rho_*([W]) \in H_4(\pi_1(W_L); \mathbb{Z}). \tag{2.2}$$

Now we can state the Fenn–Rourke Theorem.

**Theorem 18** (Fenn–Rourke Theorem). *Let $M$ be a closed, oriented 3–manifold, and let $L$ and $L'$ be two framed links in $M$. Then $L$ and $L'$ are related by a sequence of stabilizations and handle-slides if and only if there exist an orientation-preserving homeomorphism $h \colon M_L \to M_{L'}$ and an isomorphism $f \colon \pi_1(W_L) \to \pi_1(W_{L'})$ such that Diagram (2.1) commutes and $\rho_*([W]) = 0 \in H_4(\pi_1(W_L); \mathbb{Z})$.*

*Proof.* See, Fenn and Rourke [7, Theorem 6]. □

**Corollary 19.** *Let $L$ and $L'$ be two links in a closed oriented 3–manifold. Then $L$ and $L'$ have homeomorphic results of surgery if and only if they are related by a sequence of handle-slides, stabilizations and $K_3$–moves.*

*Proof.* By the $K_3$–move we add for each generator $x$ of $\pi_1(M)$ a pair $(K, K')$ to $L$ and to $L'$ with $[K] = x \in \pi_1(M)$. Let us denote the links obtained by this operation by $\widetilde{L}$ and $\widetilde{L}'$. Then $\pi_1(W_{\widetilde{L}})$ and $\pi_1(W_{\widetilde{L}'})$ are trivial and the assumptions of Theorem 18 are fulfilled. On the other hand, by Proposition 17 a $K_3$-move does not change the manifold. See [7, Theorem 8]. □

A generalization of Corollary 19 for manifolds with boundary was given by Roberts [31]. The Fenn–Rourke Theorem has been stated for manifolds with boundary by Garoufalidis–Kricker [10], but their extension only holds if the boundary is connected. In this thesis we complete these results by considering manifolds with multiple boundary components.

# Chapter 3

# Refined Kirby calculus on integral homology spheres

For integral homology spheres Habiro [15] defined a refined Kirby calculus, i.e., he proved that the Kirby moves can be refined. Here, we introduce the main results and notation of refined Kirby calculus for integral homology spheres. As mentioned in the introduction, one goal of this thesis is to give a refined Kirby calculus for admissible framed links in 3–manifolds with free abelian first homology groups. When we do so in Chapter 6 we will use the construction given in [15]. Therefore, we sketch the proof of the main result of Habiro's refined Kirby calculus.

## 3.1 Main results on refined Kirby calculus

Let $L = L_1 \sqcup \cdots \sqcup L_n$ be a link in $S^3$ with integral framing coefficients $f_i$ for $i = 1, \ldots, n$. The *linking matrix* of $L$ is defined by $A_L = \big(a_{ij}\big)$ with $a_{ij} = \mathrm{lk}(L_i, L_j)$ for $i \neq j$ and $a_{ii} = f_i$. The matrix $A_L$ is symmetric and all coefficients are integers.

Moreover, the linking matrix $A_L$ is a presentation matrix for $H_1(S^3_L, \mathbb{Z})$, i.e., $H_1(S^3_L, \mathbb{Z})$ can be identified with $\mathrm{Ker}(A_L)$. If we assume that $H_1(S^3_L, \mathbb{Z})$ is trivial, then the linking matrix has to be invertible over $\mathbb{Z}$ and therefore $\det(A_L) = \pm 1$. In this case, $A_L$ can be diagonalized and this diagonalization can be carried out by performing Kirby moves on $L$, see Section 3.2.1 for more details. Note that if $\det(A_L) = \pm 1$ the diagonal matrix $D$ has diagonal entries $\pm 1$.

A framed link $L$ is called *admissible* if it is algebraically split and $\pm 1$–framed, i.e., the linking matrix $A_L$ is diagonal with diagonal entries $\pm 1$. Thus, every integral homology sphere can be obtained by surgery along an admissible framed link in $S^3$. We denote the set of integral homology spheres by $\mathbb{Z}$ HS.

Note that neither the linking matrix nor admissibility of a link is preserved under handle-slides. Let us introduce two moves on the set of admissible links.

- A *band-slide* consists of two handle-slides of one component over the other such that they cancel algebraically, see Figure 3.1.

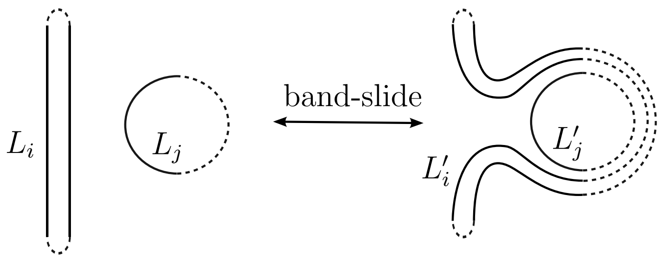



Figure 3.1: A band-slide of the component $L_i$ over $L_j$.

- A *Hoste move* is what is depicted in Figure 3.2. Let $L = L_1 \sqcup \cdots \sqcup L_n$ be an admissible framed link in $S^3$, with an unknotted component $L_i$ with framing $\pm 1$. If we do surgery along the component $L_i$ on the left hand side, we obtain a new link $L'_{L_i}$ in $S^3_{L_i} \cong S^3$ shown on the right hand side. The link $L'_{L_i}$ is again admissible. Then, the framed links $L$ and $L'_{L_i}$ are said to be related by a Hoste move.

**Theorem 20** (Habiro [15]). *Let $L$ and $L'$ be two admissible framed links in $S^3$. Then the following are equivalent.*

*(i) $L$ and $L'$ have orientation-preserving homeomorphic results of surgery,*

*(ii) $L$ and $L'$ are related by a sequence of band-slides and stabilizations,*

*(iii) $L$ and $L'$ are related by a sequence of Hoste moves.*

Using the techniques developed by Habiro in [15] a similar result was proved by Otmani [30]. Otmani considered links whose linking matrices are of the form $H^n = \oplus_n H$ where $\oplus$ denotes block sum and $H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

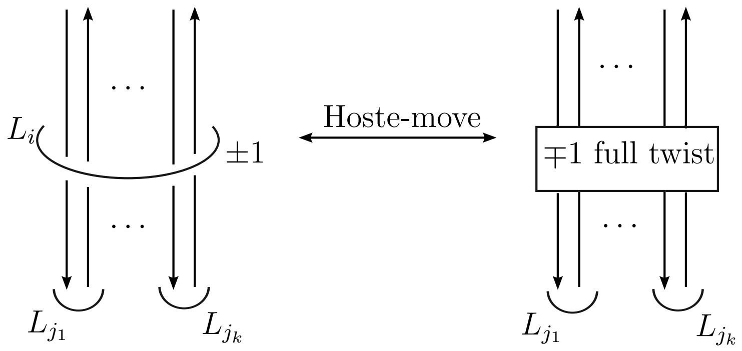



Figure 3.2: A Hoste-move along the unknotted component $L_i$.

Links of this form represent integral homology spheres for which a certain invariant is zero, i.e., with vanishing Rochlin invariant. Fujiwara [8] extended Theorem 20 to 3–manifolds with first homology group of odd prime order.

## 3.2 The Kirby moves algebraically

For a fixed integer $n \geq 0$ we denote by $\mathcal{L}_{M,n} = \mathcal{L}$ the set of $n$–component, oriented, ordered framed links in a 3–manifold $M$. A link $L \in \mathcal{L}$ is denoted by $L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_n$. Define a set $\mathcal{E}$ of elementary moves on links in $\mathcal{L}$ as follows:

- $p_{i,j}$*–move*: exchanges the component $L_i$ with $L_j$,
- $q_i$*–move*: reverses the orientation of $L_i$,
- $w^{\epsilon}_{i,j}$*–move*: is a handle-slide of $L_i$ over $L_j$ with the orientations as in Figure 3.3 for $\epsilon = \pm 1$.

Let $L$ and $L'$ be links in $\mathcal{L}$. If $L'$ is obtained from $L$ by an elementary move $e \in \mathcal{E}$ we write $L \overset{e}{\to} L'$. If $e$ is either a $p_{i,j}$–move or a $q_i$–move the resulting link $L'$ is unique. But for $e = w^{\epsilon}_{i,j}$ there are in general infinitely many distinct $L'$ satisfying $L \overset{e}{\to} L'$. If $L'$ is obtained from $L$ through a sequence $S$ of elementary moves $e_i \in \mathcal{E}$ for $i = 1, \ldots, p$ we write

$$S : L = L^0 \overset{e_1}{\to} L^1 \overset{e_2}{\to} \cdots \overset{e_p}{\to} L^p = L',$$

or, in short $S : L \to L'$. The composition of two sequences of elementary moves $S : L^0 \overset{e_1}{\to} L^1 \overset{e_2}{\to} \cdots \overset{e_p}{\to} L^p$ and $\tilde{S} : \tilde{L}^0 \overset{\tilde{e_1}}{\to} \tilde{L}^1 \overset{\tilde{e_2}}{\to} \cdots \overset{\tilde{e_q}}{\to} \tilde{L}^q$ with $L^p = \tilde{L}^0$

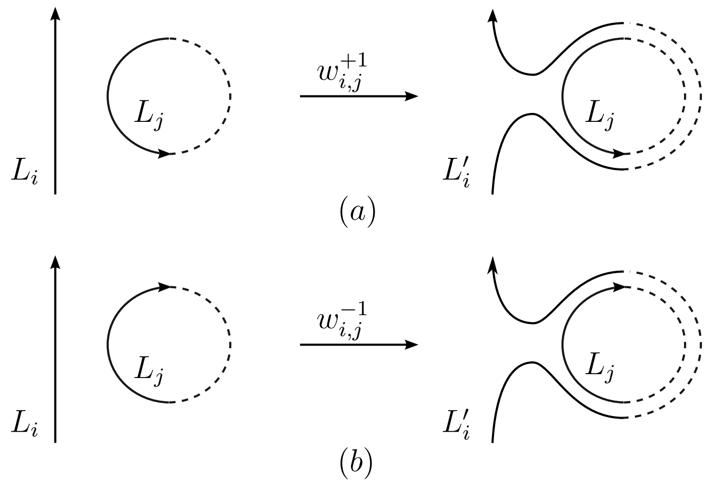



Figure 3.3: (a) A $w_{i,j}^{+1}$-move. (b) A $w_{i,j}^{-1}$-move.

is given by

$$\tilde{S}S : L^0 \xrightarrow{e_1} L^1 \xrightarrow{e_2} \cdots \xrightarrow{e_p} L^p = \tilde{L}^0 \xrightarrow{\tilde{e_1}} \tilde{L}^1 \xrightarrow{\tilde{e_2}} \cdots \xrightarrow{\tilde{e_q}} \tilde{L}^q.$$

Denote by $\mathcal{S}_n$ the set of sequences of elementary moves on $\mathcal{L}_n$. To each sequence of elementary moves in $\mathcal{S}_n$ we associate a matrix in $GL(n;\mathbb{Z})$ as follows. Let $E_{i,j}$ be the matrix whose $(i,j)$–entry is 1 and all other entries are 0. We denote by $I_n$ the identity matrix of size $n$. Consider the following matrices:

$$\begin{aligned} P_{i,j} &= I_n - E_{i,i} - E_{j,j} + E_{i,j} + E_{j,i}, \\ Q_i &= I_n - 2E_{i,i}, \\ W_{i,j}^{\pm 1} &= I_n \pm E_{i,j}. \end{aligned}$$

They generate $GL(n;\mathbb{Z})$ and we define a map $\varphi : \mathcal{S}_n \to GL(n;\mathbb{Z})$ by

$$\varphi(L \xrightarrow{p_{i,j}} L') = P_{i,j}, \quad \varphi(L \xrightarrow{q_i} L') = Q_i, \quad \varphi(L \xrightarrow{w_{i,j}^{\pm 1}} L') = W_{i,j}^{\pm 1}.$$

For a sequence $S : L^0 \xrightarrow{e_1} L^1 \xrightarrow{e_2} \cdots \xrightarrow{e_p} L^p$ we set

$$\varphi(S) = \varphi(L^{p-1} \xrightarrow{e_p} L^p) \cdots \varphi(L^1 \xrightarrow{e_2} L^2)\varphi(L^0 \xrightarrow{e_1} L^1).$$

Alternatively, the map $\varphi$ can also be defined in a functorial way, see [15] for more details. The matrix $\varphi(S)$ is called the *associated matrix* to the sequence $S \in \mathcal{S}_n$. Now we can state the Main Lemma of [15].

**Lemma 21** (Main Lemma [15])**.** *If a sequence $S : L \to L'$ satisfies $\varphi(S) = I_n$, then $L$ and $L'$ are related by a sequence of band-slides.*

*Proof.* The proof is given in [15, Section 3]. It is a non-constructive existence proof given in algebraic terms. □

### 3.2.1 Sketch of proof of Theorem 20

For any sequence $S : L \to L'$ of elementary moves between two links $L, L' \in \mathcal{L}_{S^3,n}$ we have the following relation on the linking matrices.

$$A_{L'} = \varphi(S) A_L \varphi(S)^t, \tag{3.1}$$

where $\varphi(S)^t$ denotes the transpose of $\varphi(S)$ [15, Lemma 2.2],[18]. Let us define a matrix $I_{p,q} = I_p \oplus (-I_q)$ where $\oplus$ denotes the block sum. For an admissible framed link $L \in \mathcal{L}_{S^3,n}$ we have $A_L = I_{p,q}$ for some $p, q \in \mathbb{N}$ with $p + q = n$. Therefore, if $S \in \mathcal{S}_n$ is a sequence between two $n$-component, admissible framed links its associated matrix $\varphi(S)$ lies in the subgroup

$$O(p, q; \mathbb{Z}) = \{T \in GL(p + q; \mathbb{Z}) \mid T I_{p,q} T^t = I_{p,q}\}$$

of $GL(n; \mathbb{Z})$, with $n = p + q$. By a theorem of Wall [34] the group $O(p, q; \mathbb{Z})$ is generated by the elements $P_{i,j}, Q_i$ and the matrix

$$D = \begin{pmatrix} 1 & 1 & \mathbf{0} & -1 & 0 & \mathbf{0} \\ -1 & 1 & \mathbf{0} & 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{p-2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -1 & 0 & \mathbf{0} & 1 & 1 & \mathbf{0} \\ 0 & 1 & \mathbf{0} & -1 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -I_{q-2} \end{pmatrix}.$$

**The main steps to prove Theorem 20**

- Step 1: Show that for every element $T \in O(p, q; \mathbb{Z})$ with $p+q = n$, there exists a sequence $S$ between the $n$–component unlink $U_n$ and itself such that:

  $$\varphi(U_n \xrightarrow{S} U_n) = T.$$

  It is enough to show that such a sequence exists for every generator, i.e., for $P_{i,j}, Q_i$ and $D$. For $P_{i,j}$ and $Q_i$ we can choose the $p_{i,j}$– and $q_i$–move, i.e., we can reorder the link or change the orientations. For the matrix $D$ there is more work to do. Consider for example the following decomposition of $D$ where $n = 4$.

$$D = W_{2,1}^{-1} W_{3,1}^{-1} W_{2,4} W_{3,4} W_{4,3}^{-1} W_{1,3}^{-1} W_{4,2} W_{1,2}.$$

It is possible to find a sequence of handle-slides on the 4–component unknot $U_4$ corresponding to this decomposition, see [15, Figure 7].

- Step 2: Let $L$ and $L'$ be two admissible framed links in $\mathcal{L}_{S^3,n}$ related by a sequence $S \in \mathcal{S}_n$. Stabilize $L$ and $L'$ sufficiently many times with $\pm1$-framed unknots and denote the stabilized links by $\hat{L}, \hat{L}'$. One can show that a crossing change can be obtained by using band-slides along a $\pm1$-framed unknot. Thus, we can use the added $\pm1$-framed unknots to unknot the original components of $L$ in $\hat{L}$ by a sequence of band-slides, while the added unknots get knotted. Hence, we obtain a link $\hat{L}^{\#}$ that has unknotted components at the places we need. By Step 1 we can then find a sequence $S^{\#}\colon \hat{L}^{\#} \to \hat{L}^{\#}$ with $\varphi(S^{\#}) = \varphi(\hat{S})^{-1}$. See Figure 3.4 for a schematic overview.

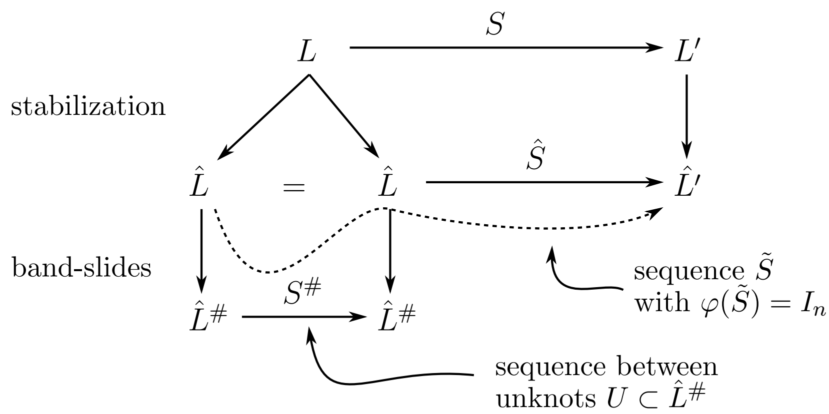



Figure 3.4: Sketch of the proof, Step 2.

- Step 3: Apply Habiro's Main Lemma 21 to the composition $\tilde{S}\colon \hat{L} \to \hat{L}'$ defined in Step 2.

### Main difficulties for generalization

Let $M$ be a 3–manifold with free abelian first homology group and $L$ a link in $M$.

- In Step 1: In order to find the sequence $S\colon U_n \to U_n$ for an unknot $U_n \in M$ as described in above, it is essential that we consider unknots $U_n$ that are homotopically trivial in $M$. This is always the case in $S^3$ but might not be the case in $M$.

- In Step 2: In $S^3$ we could reorder the components by band-slides. As long as the links are considered to be null-homotopic, this can still be done. Otherwise we can only unknot the components by using band-slides but this does not change their homotopy type. Thus we have to modify Step 1. For null-homologous links in $M$ we solve this problem by allowing an additional move.

# Chapter 4

# Clasper calculus

In the first section of this chapter the definition of a clasper is given. Claspers induce an equivalence relation on 3–manifolds which we study in Section 4.2. Moreover, to see why claspers are important, we give an overview of the Goussarov-Habiro filtration and the definition of finite type invariants. Finally, we can describe the IHX-link with claspers. The IHX-link is a key ingredient of the results in Chapter 6.

## 4.1 Introduction to claspers

Claspers were independently introduced by Goussarov [12] and Habiro [14] in the study of finite type invariants for 3–manifolds. Here, we follow the notation and conventions of Habiro [14].

A *graph clasper* in a 3–manifold $M$ is a compact connected surface $G$ embedded in the interior of $M$ that splits into three kinds of subsurfaces: *edges, nodes* and *leaves*. An *edge* is a band, a *node* is a disk and a *leaf* is an annulus. Leaves and nodes are called *constituents*. Moreover, every leaf is connected to exactly one edge, every node is connected to three edges and every edge connects two distinct constituents or it connects a node with itself. We draw a graph clasper by using the blackboard framing, see Figure 4.1.

The *degree* of a graph clasper is defined as the number of its nodes. A graph clasper of degree $k$ is also called a *$Y_k$–graph*. The graph clasper shown in Figure 4.1 is a $Y_4$–graph. A *basic clasper* is a graph clasper with only one edge and two leaves as shown in Figure 4.2.

To a graph clasper we associate a framed link as follows. First, every graph clasper is split into a union of basic claspers by replacing each node with three leaves forming a Borromean ring as in Figure 4.3. Then, to each basic clasper we associate a 2–component framed link as shown in 4.4 ($b$).

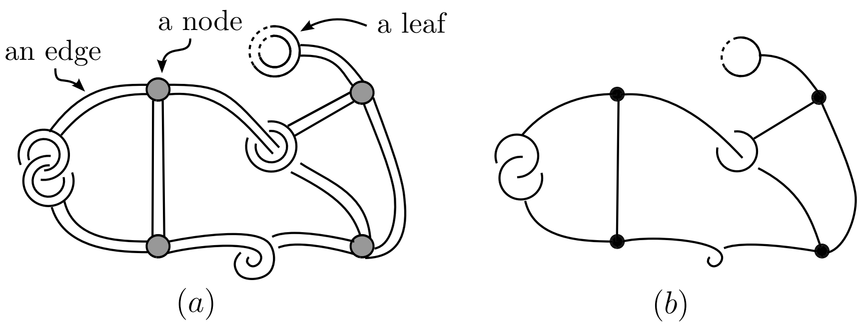



Figure 4.1: (*a*) A graph clasper. (*b*) The same graph clasper using the blackboard framing.

Figure 4.2: A basic clasper and its presentation using blackboard framing.

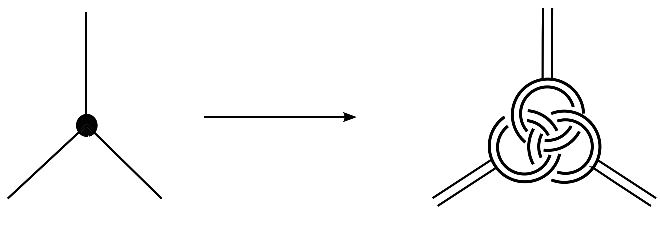


Figure 4.3: Replacing a node by three leaves.


Figure 4.4: (*a*) A basic clasper. (*b*) The blackboard framed link associated to a basic clasper.

*Surgery on a graph clasper* $G$ in a 3–manifold $M$ is defined as performing surgery along the associated framed link. The manifold obtained by surgery on $G$ is denoted by $M_G$. A $Y_k$*–surgery* on $M$ is the surgery along some $Y_k$–graph in M. In Figure 4.5 we shows how to replace a $Y_1$–graph by its associated surgery link.

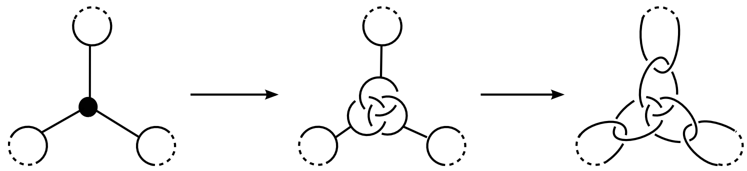
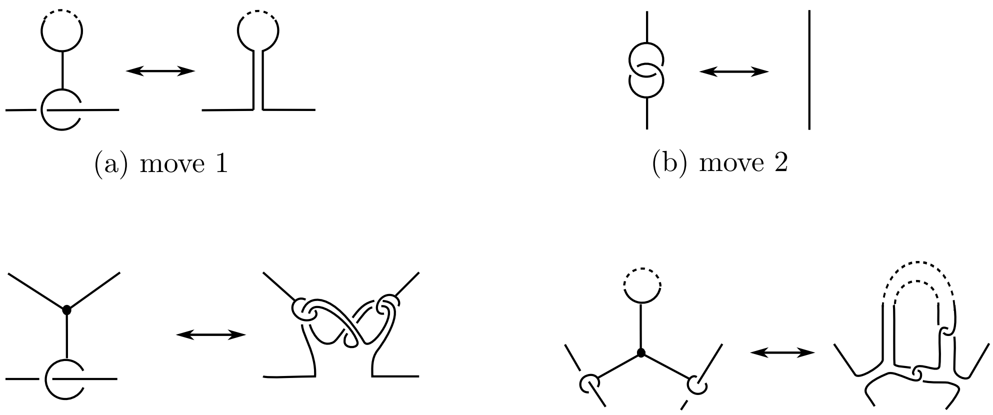


Figure 4.5: A $Y_1$–graph and its associated surgery link.

We define the $Y_k$*–equivalence class* $Y_k(M)$ of a 3–manifold $M$ as the set of all 3–manifolds that can be obtained from $M$ by a sequence of $Y_k$–surgeries. The $Y_k$–equivalence is called $(k-1)$–equivalence in [12] and $A_k$–equivalence in [14].

Clasper calculus is the study of moves on graph claspers such that surgery along the graph claspers produces the same manifold. Clasper calculus also applies if the 3–manifold $M$ (possibly with non-empty boundary) contains a framed oriented tangle $\gamma \subset M$ whose boundary (if any) corresponds to marked points on the surface $\partial M$. If $\partial M = \emptyset$ then $\gamma$ is a link. Let us give some clasper moves from [14, Figure 9].



(a) move 1

(b) move 2

(c) move 9

(d) move 10

Figure 4.6: Some clasper moves.

These clasper moves are designed so that the following proposition holds.

**Proposition 22.** *[14, Proposition 2.7] Let $G$ and $G'$ be two graph claspers related by a sequence of the moves in Figure 4.6. Then there exists an orientation-preserving homeomorphism $h : M_G \to M_{G'}$.*

A $Y_k$–graph is called a $Y_k$*–tree* if its union of nodes and edges is simply connected. The "move 2" of Figure 4.6b allows us to split any graph into trees. Thus, it is enough to consider $Y_k$–trees to generate the $Y_k$–equivalence.

## 4.2 The $Y_k$–equivalence

A $Y_1$–surgery is equivalent to Borromean surgery introduced by Matveev [26]. Matveev showed that two closed 3–manifolds are Borromean surgery equivalent if and only if there is an isomorphism on their first homology groups that induces isomorphic linking pairings[1]. Thus, every integral homology sphere is $Y_1$–surgery equivalent to $S^3$.

The $Y_k$–equivalences classes have the following property.

**Proposition 23.** *For $1 \leq k \leq k'$, any two $Y_{k'}$–equivalent 3–manifolds are also $Y_k$–equivalent, i.e.,*

$$Y_1(M) \supset Y_2(M) \supset Y_3(M) \supset \cdots .$$

*Proof.* Reduce the degree of a clasper by move 9 of Figure 4.6c, or see [14, Proposition 3.7]. □

For $M = S^3$ we get a filtration on the space of integral homology spheres $\mathbb{Z}\,\mathrm{HS}$, since $Y_1(S^3) = \mathbb{Z}\,\mathrm{HS}$.

Cochran, Gerges and Orr [3] introduced a related filtration using the notion of $k$–surgery equivalence. Recall that a framed link in a 3–manifold $M$ is called *admissible* if it is null-homologous, algebraically split and $\pm1$-framed, i.e., its linking matrix is diagonal with diagonal entries $\pm1$. Two 3–manifolds $M$ and $M'$ are said to be *$k$–surgery equivalent* if there exist an admissible framed link $L$ in $M$ with $M_L \cong M'$ such that the homotopy class of each component of $L$ lies in the $k$–th term of the lower central series of the fundamental group $\pi := \pi_1(M)$. The subgroups corresponding to the lower central series of $\pi$ are inductively defined as:

$$\pi = \Gamma_1\pi \supset \Gamma_2\pi \supset \cdots \supset \Gamma_k\pi \supset \Gamma_{k+1}\pi \supset \cdots$$

[1]The linking pairing is a specific pairing on torsion subgroups of the first homology group induced by the linking matrix.

where $\Gamma_{i+1}\pi := [\Gamma_i\pi, \pi]$. Then the implications

$$Y_1\text{–equivalence} \longleftarrow 2\text{–surgery equivalence} \longleftarrow Y_2\text{–equivalence},$$

and

$$k\text{–surgery equivalence} \longleftarrow Y_{2k-2}\text{–equivalence},$$

hold for $k \geq 2$, see [14, Section 8.4.3].

Note that any integral homology sphere is $k$–surgery equivalent to $S^3$ for all $k \geq 2$, while the $Y_k$–surgery equivalence becomes strictly finer as $k$ increases.

Clasper calculus can also be considered as topological commutator calculus for 3–manifolds. For a connected, compact, oriented surface $\Sigma$ the Torelli group is the group of self-homeomorphisms of $\Sigma$ up to homotopy inducing the identity on homology.

**Proposition 24** ([14]). *Two 3–manifolds $M$ and $M'$ are $Y_k$–equivalent if and only if $M'$ is obtained from $M$ by cutting $M$ open along an embedded connected, compact, oriented surface $\Sigma \subset M$ and regluing it by an element in the $k$–th term of the lower central series of the Torelli group.*

*Proof.* For a proof see [24, Appendix]. □

### 4.2.1 Homology cylinders

Both, Goussarov [12] and Habiro [14] considered also 3–manifolds with boundaries. They studied cobordisms between surfaces and focused particularly on homology cylinders.

Let $\Sigma_{g,n}$ be a surface of genus $g$ with $n \geq 0$ boundary components. A *homology cobordism* from $\Sigma_{g,n}$ to $\Sigma_{g,n}$ is a pair $(M, m)$ where $M$ is a 3–manifold and $m : \partial(\Sigma_{g,n} \times [-1, 1]) \to \partial M$ is an orientation-preserving homeomorphism such that both inclusions $m_\pm : \Sigma_{g,n} \times \{\pm 1\} \to M$ induce isomorphisms on homology $\mathrm{H}_*(\Sigma_{g,n}) \overset{\cong}{\to} \mathrm{H}_*(M)$.
If $(m_-)^{-1} \circ m_+$ is the identity map on homology, the pair $(M, m)$ is called a *homology cylinder*.
Two homology cobordisms $(M, m), (M', m')$ are called *homeomorphic* if there is an orientation-preserving homeomorphism $f : M \to M'$ such that $f|_{\partial M} \circ m = m'$. The set of homeomorphism classes of homology cobordisms and homology cylinders is denoted by $\mathcal{C}(\Sigma_{g,n})$ and $\mathcal{IC}(\Sigma_{g,n})$, respectively. The

composition of two homology cobordisms $M, M' \in \mathcal{C}(\Sigma_{g,n})$ is defined by gluing the "bottom" of $M'$ to the "top" of $M$, i.e.,

$$M \circ M' = M \cup_{m_+ \circ (m'_-)^{-1}} M'$$

with the obvious parametrization of $\partial(M \circ M')$. Both $\mathcal{C}(\Sigma_{g,n})$ and $\mathcal{IC}(\Sigma_{g,n})$ are stable under composition and therefore monoids.

**Proposition 25.** *The monoid of homology cylinders $\mathcal{IC}(\Sigma_{g,n})$ equals the $Y_1$–equivalence class of the trivial cobordism $(\Sigma_{g,n} \times I)$, i.e.,*

$$\mathcal{IC}(\Sigma_{g,n}) \cong Y_1(\Sigma_{g,n} \times I).$$

*Proof.* This surgery characterization was stated in [14], a proof is given in [24, 13]. □

Let us define the set of *special homology cylinders* by

$$s\mathcal{IC}(\Sigma_{g,n}) := \{(\Sigma_{g,n} \times I)_L \mid L \text{ null-homologous, admissible framed link } \}.$$

Thus, $s\mathcal{IC}(\Sigma_{g,n})$ is the 2–surgery equivalence class of the trivial cobordism $(\Sigma_{g,n} \times I)$ and we have the following proper inclusions:

$$\mathcal{C}(\Sigma_{g,n}) \supset \mathcal{IC}(\Sigma_{g,n}) \supset s\mathcal{IC}(\Sigma_{g,n}).$$

Moreover, there is a close connection between homology cobordisms and the mapping class group. For a survey on this interesting subject see [16].

## 4.3 The Goussarov-Habiro filtration

In this subsection we sketch a theory of finite type invariants in terms of claspers [12] [14]. The notion of finite type invariants has been introduced for links as well as for 3–manifolds to study their set of invariants. The theory of finite type invariants is one of the main motivation to use clasper calculus.

An *allowable graph scheme* $S$ is a set of disjoint graph claspers $S = \{G_1, \ldots, G_m\}$ in a 3–manifold $M$ where no $G_i$ is a basic clasper. The *degree* of $S$ is the sum of the degrees of its elements. Furthermore, for a closed 3–manifold $M$ define $\mathcal{M}(M)$ as the free abelian group generated by the elements of the $Y_1$–equivalence class of $M$. For an allowable graph scheme $S$, define $[M, S] \in \mathcal{M}(M)$ by

$$[M, S] = \sum_{S' \subset S} (-1)^{|S'|} M_{S'}$$

where the sum is taken over all subsets $S'$ of $S$, $|S'|$ is the number of elements of $S'$ and $M_{S'}$ denotes the result of surgery on all graph claspers in $S'$ in $M$. The *Goussarov-Habiro filtration* is the descending filtration

$$\mathcal{M}(M) = \mathcal{M}_1(M) \supset \mathcal{M}_2(M) \supset \cdots$$

where $\mathcal{M}_k(M)$ is the subgroup generated by elements $[M, S_k]$ for an allowable graph scheme $S_k$ of degree $k$.

An invariant $f : \mathcal{M}(M) \to A$ with target space an abelian group $A$ is called of *finite type of degree at most* $d$ (in the Goussarov-Habiro-sense) if it vanishes on $\mathcal{M}_{d+1}(M)$.
One example of an invariant of finite type of degree $2k$ in this sense is the degree $k$ term $\Omega_k$ of the LMO–invariant of closed 3–manifolds defined by Le, Murakami and Ohtsuki in [21].

There are several theories of finite type invariants. The first theory of finite type invariants for integral homology spheres $\mathbb{Z}\,\mathrm{HS}$ was introduced by Ohtsuki [29] and is based on surgery along admissible framed links in $S^3$. Ohtsuki's approach was extended to more general 3–manifolds by Cochran and Melvin [4]. The different filtrations have been studied by Habiro in [14, Section 8.4.3] as well as by Garoufalidis, Goussarov and Polyak in [9] where they also compare other filtrations. For integral homology spheres the Goussarov-Habiro filtration coincides with the Ohtsuki filtration over $\mathbb{Z}[1/2]$.

## 4.4 Jacobi-diagrams, AS- and IHX-relations

Finally, we introduce Jacobi-diagrams to give a characterization of the graded quotient $\mathcal{G}_k(M) = \mathcal{M}_k(M)/\mathcal{M}_{k+1}(M)$. Write $[M, G] =_k [M, G']$ if $[M, G] - [M, G'] \in \mathcal{M}_{k+1}(M)$.

In the graded quotient $\mathcal{G}_k(M)$ the following relations hold.

- AS-relation: $[M, G] =_k -[M, G']$, where $G$ and $G'$ differ locally as depicted in Figure 4.7.


Figure 4.7: The graph claspers $G$ and $G'$.

- IHX-relation: $[M, G_I] - [M, G_H] + [M, G_X] =_k 0$, where $G_I, G_H, G_X$ differ locally as in Figure 4.8.

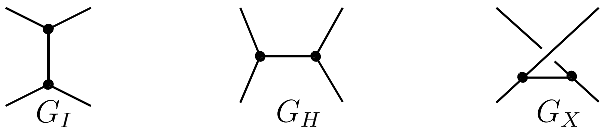



Figure 4.8: The graph claspers $G_I, G_H$ and $G_X$.

For a proof see [9, Corollary 4.6, Theorem 4.11].

A *Jacobi diagram* is a trivalent graph, see Figure 4.9 (a), with a cyclic ordering of the incident edges in each vertex. The *degree* of a Jacobi diagram is half the number of its vertices. Denote by $\mathcal{A}_k$ the rational vector space generated by all degree $k$ Jacobi diagrams modulo the AS and IHX relation given in Figure 4.7 and Figure 4.8. We can embed any Jacobi diagram $\Gamma$ into $S^3$. By replacing each edge as in Figure 4.6b we can associate an allowable graph scheme $S(\Gamma)$ to each $\Gamma$, see Figure 4.9 (b).

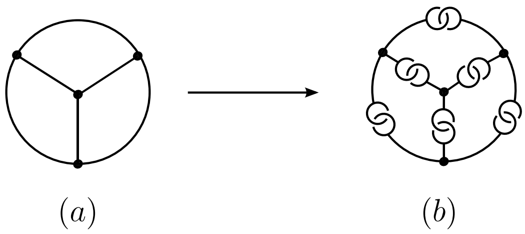



Figure 4.9: (a) A Jacobi diagram $\Gamma$ of degree 2. (b) Its associated allowable graph scheme $S(\Gamma)$ of degree 4.

This defines a map :

$$\begin{aligned}\Phi\colon \mathcal{A}_k &\to \mathcal{G}_{2k}(S^3)\\ \Gamma &\mapsto [S^3, S(\Gamma)]_k\end{aligned}$$

where $[S^3, S(\Gamma)]_k$ denotes the equivalence class of $[S^3, S(\Gamma)]$ in $\mathcal{G}_k(S^3)$. It is shown in [9, 14, 20] that for $k$ odd $\mathcal{G}_k(S^3) = 0$ and the map $\Phi\colon \mathcal{A}_k \to \mathcal{G}_{2k}(S^3)$ is an isomorphism for any $k$.

The IHX-relation described here is also closely related to the Jacobi identity, the main relation imposed on the binary operation of a Lie algebra.

### 4.4.1 IHX-clasper and IHX-link

Consider the thickened punctured disk

$$V_g := (D^2 \setminus \{x_1, \dots, x_g\}) \times I$$

where $x_1, \dots x_g \in D^2$ and $I = [0, 1]$. The trivial string link $x_1 \times I, \dots, x_g \times I \in D^2 \times I$ represents $V_g$. Moreover $V_g$ is orientation-preserving homeomorphic to the 3-dimensional handlebody of genus $g$.

We define the *IHX-clasper* consisting of three $Y_2$–graphs in $V_4$ to be the clasper depicted in Figure 4.10. Replacing the three $Y_2$-graphs $T_1, T_2$ and $T_3$

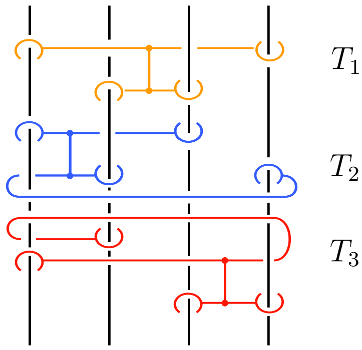



Figure 4.10: The IHX-clasper in $V_4$.

by the associated framed links as shown in Figure 4.11 yields a 6-component framed link $L_{IHX} \subset V_4$ which we call the *IHX-link*. Here the blackboard framing is assumed.

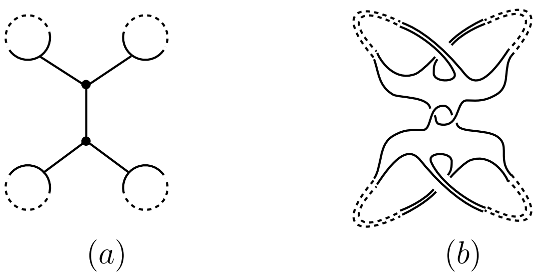



Figure 4.11: (a) $Y_2$-graph. (b) Associated framed link.

For any embedding of $V_4$ into a 3-manifold $M$

$$f : V_4 \hookrightarrow M,$$

we can show that there is a boundary-preserving diffeomorphism $h\colon M_{L_{IHX}} \to M$. A similar result was proven in [6, 5] by using different embeddings of the three $Y_2$-graphs into $V_4$.

## 4.5 Some clasper relations

For the reader's convenience we compare segments of a clasper depicted as surfaces and as blackboard framed graphs in Figure 4.12 . A $\pm 1$ twist of a band is labelled $s^{\pm 1}$. Moreover, the effect on the corresponding surgery link is shown.

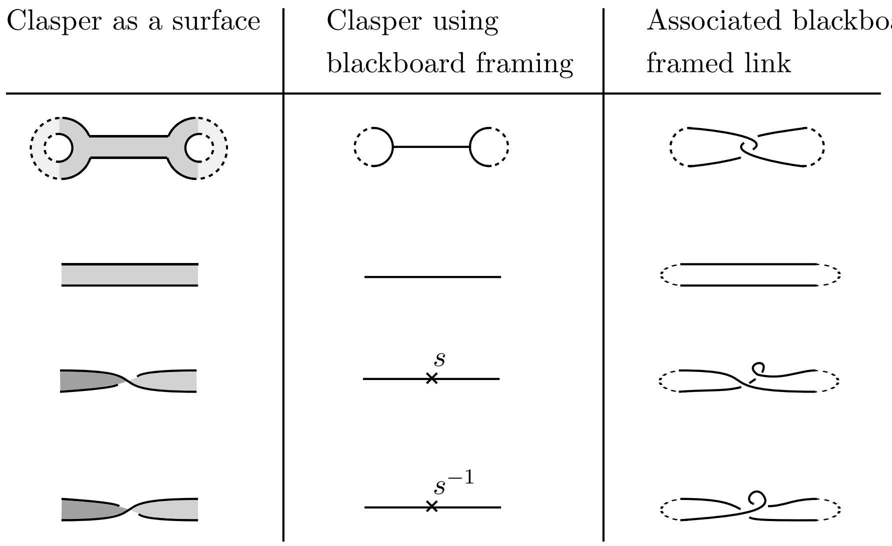



Figure 4.12: Clasper segments: depicted as a surface, as a blackboard framed graph and the corresponding surgery link.

In Figure 4.13 we show some clasper relations that are useful in computations. The top-right relation in Figure 4.13 is move 2 from Figure 4.6b.

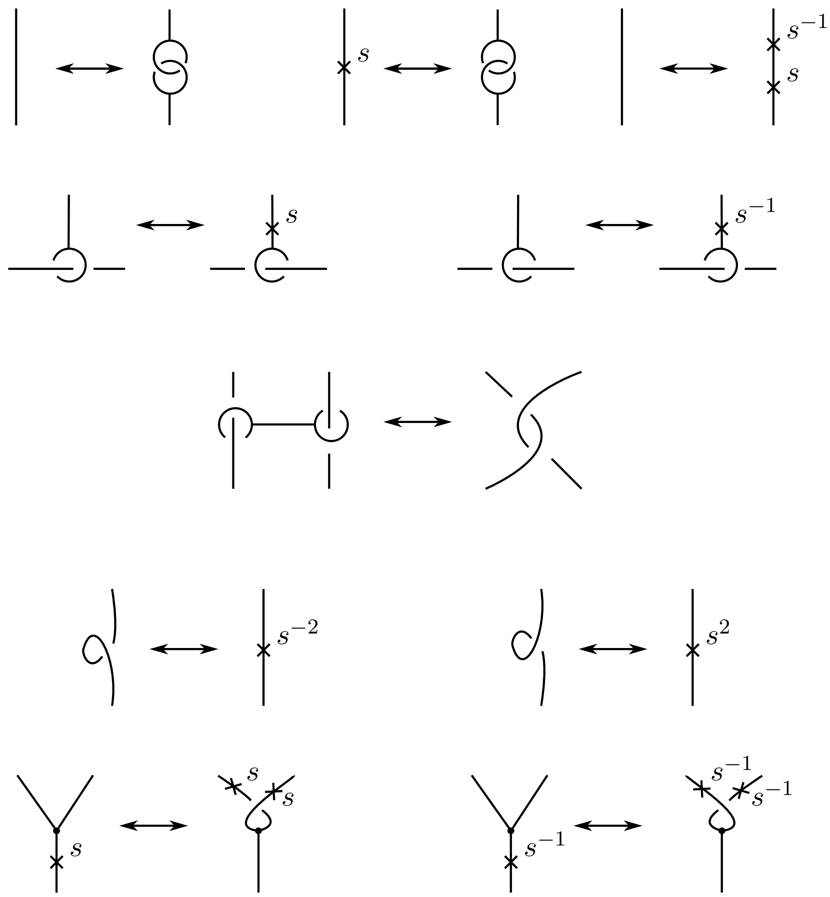



Figure 4.13: Some useful clasper tricks.

# Part II

# Results

# Chapter 5

# On Kirby calculus for null-homotopic framed links in 3–manifolds

This article is a joint work with K. Habiro. The version presented herein is the one that will appear in the journal of Algebraic & Geometric Topology with different page numbering and with original enumerations otherwise.

# On Kirby calculus for null-homotopic framed links in 3–manifolds

KAZUO HABIRO
TAMARA WIDMER

Kirby proved that two framed links in $S^3$ give orientation-preserving homeomorphic results of surgery if and only if these two links are related by a sequence of two kinds of moves called stabilizations and handle-slides. Fenn and Rourke gave a necessary and sufficient condition for two framed links in a closed, oriented 3–manifold to be related by a finite sequence of these moves.

The purpose of this paper is twofold. We first give a generalization of Fenn and Rourke's result to 3–manifolds with boundary. Then we apply this result to the case of framed links whose components are null-homotopic in the 3–manifold.

57M25, 57M27

## 1 Introduction

In 1978, Kirby [10] proved that two framed links in $S^3$ have homeomorphic result of surgery if and only if they are related by a sequence of two kinds of moves called stabilizations and handle-slides. This result enables one to construct a 3–manifold invariant by constructing a link invariant which is invariant under these moves. Fenn and Rourke [5] generalized Kirby's theorem to framed links in closed 3–manifolds, and Roberts [11] generalized it to framed links in 3–manifolds with boundary.

Fenn and Rourke [5] also considered the equivalence relation on framed links in an arbitrary closed, oriented 3–manifold generated by stabilizations and handle-slides. Here we state Fenn and Rourke's theorem, leaving some details to the original paper [5]. Let $M$ be a closed, oriented 3–manifold. For a framed link $L$ in $M$, we will denote by $W_L$ the 4–manifold obtained from $M \times I$ by attaching 2–handles along $L \times \{1\} \subset \partial(M \times I)$ in a way determined by the framing. Note that $W_L$ is a cobordism between $M$ and $M_L$, where $M_L$ denotes the 3–manifold obtained from $M$ by surgery along $L$. The inclusions $M_L \hookrightarrow W_L \hookleftarrow M$ induce surjective homomorphisms

$$\pi_1(M_L) \twoheadrightarrow \pi_1(W_L) \twoheadleftarrow \pi_1(M).$$

The kernel of the homomorphism $\pi_1(M) \to \pi_1(W_L)$ is normally generated by the homotopy classes of components of $L$.

**Theorem 1.1** (Fenn–Rourke [5]) *Let $M$ be a closed, oriented 3–manifold, and let $L$ and $L'$ be two framed links in $M$. Then $L$ and $L'$ are related by a sequence of stabilizations and handle-slides if and only if there exist an orientation-preserving homeomorphism $h\colon M_L \to M_{L'}$ and an isomorphism*

$$f\colon \pi_1(W_L) \to \pi_1(W_{L'}),$$

*such that the diagram*

$$\begin{tikzcd} \pi_1(M_L) \arrow[r, "h_*"] \arrow[d] & \pi_1(M_{L'}) \arrow[d] \\ \pi_1(W_L) \arrow[r, "f"] & \pi_1(W_{L'}) \\ & \pi_1(M) \arrow[ul] \arrow[u] \end{tikzcd} \tag{1}$$

*commutes and we have $\rho_*([W]) = 0 \in H_4(\pi_1(W_L), \mathbb{Z})$. Here*

- *$W$ is the closed 4–manifold obtained from $W_L$ and $W_{L'}$ by gluing along their boundaries using $\mathrm{id}_M$ and $h$,*
- *$[W] \in H_4(W, \mathbb{Z})$ is the fundamental class, and*
- *$\rho_*\colon H_4(W, \mathbb{Z}) \to H_4(\pi_1(W_L), \mathbb{Z})$ is induced by a map $\rho\colon W \to K(\pi_1(W_L), 1)$ obtained by gluing natural maps from $W_L$ and $W_{L'}$ to $K(\pi_1(W_L), 1)$.*

*See [5] for more details.*

One of the main results of the present paper, Theorem 2.2, is a generalization of Theorem 1.1 to 3–manifolds with boundary. (A generalization of Theorem 1.1 to 3–manifolds with boundary has been stated in [6], but unfortunately the statement in [6] is not correct for 3–manifolds with more than one boundary components.)

An obstruction to making Theorems 1.1 and 2.2 useful is the homological condition $\rho_*([W]) = 0$. Given framed links $L, L'$ in $M$ as in Theorems 1.1 and 2.2, it is not always easy to see whether we have $\rho_*([W]) = 0$ or not. However, if $H_4(\pi_1(W_L), \mathbb{Z}) = 0$, then clearly we have $\rho_*([W]) = 0$.

A large class of groups with vanishing $H_4(-, \mathbb{Z})$ is the 3–manifold groups. It seems to have been well known for a long time that if $M$ is a compact, connected, oriented 3–manifold, then we have $H_4(\pi_1(M), \mathbb{Z}) = 0$ (see Lemma 3.3). So, if the components of the framed links $L$ and $L'$ in $M$ are null-homotopic, then since $\pi_1(W_L) \cong \pi_1(M)$ is a 3–manifold group, we have $H_4(\pi_1(W_L), \mathbb{Z}) = 0$ and $\rho_*([W]) = 0$. Thus, for null-homotopic framed links, we do not need the condition $\rho_*([W]) = 0$, see Theorem 3.1.

Cochran, Gerges and Orr [3] studied surgery along null-homologous framed links with diagonal linking matrices with diagonal entries $\pm 1$, and also surgery along more special classes of framed links. This includes null-homotopic framed links with diagonal linking matrices with diagonal entries $\pm 1$. Let us call such a framed link $\pi_1$*–admissible*. Surgery along a $\pi_1$–admissible framed link $L$ in a 3–manifold $M$ gives a manifold $M_L$ whose fundamental group is "very close" to that of $M$. In [3] it is proved that, for all $d \geq 1$, we have $\pi_1(M_L)/\Gamma_d\pi_1(M_L) \cong \pi_1(M)/\Gamma_d\pi_1(M)$, where for a group $G$, $\Gamma_d G$ denotes the $d$th lower central series subgroup of $G$.

For $\pi_1$–admissible framed links in a 3–manifold, we can combine Theorem 3.1 with Proposition 4.1 proved by the first author [8] to obtain a refined version of Theorem 3.1, see Theorem 4.2. This theorem gives a necessary and sufficient condition for two $\pi_1$–admissible framed links in $M$ to be related by a sequence of stabilizations and *band-slides* [8], which are pairs of algebraically cancelling handle-slides, see Section 4.

We apply Theorem 4.2 to surgery along null-homotopic framed links in cylinders over surfaces. Surgery along a $\pi_1$–admissible framed link in a cylinder over a surface gives a homology cylinder of a special kind.

The organization of the rest of the paper is as follows. In Section 2, we introduce some notations and preliminary facts, and then state and prove the generalization of Fenn and Rourke's theorem to 3–manifolds with boundary. In Section 3, we focus on the case of null-homotopic framed links. In Section 4, we consider $\pi_1$–admissible framed links. In Section 5, we give an example which illustrates the conditions needed in Theorem 2.2.

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## 2 Generalization of Fenn and Rourke's Theorem

In this section we state and prove a generalization of Theorem 1.1 to 3–manifolds with nonempty boundary. We start by giving necessary notations which are used throughout this paper. Then we introduce the conditions under which Theorem 1.1 holds for manifolds with boundary and give the statement and the proof of our generalization of Theorem 1.1. Our construction mainly follows [5] and borrows some ideas also from [6].

Let $M$ be a compact, connected, oriented 3–manifold, possibly with nonempty boundary.

A *framed link* $L = L_1 \cup \cdots \cup L_l$ in $M$ is a link (i.e., disjoint union of finitely many embedded circles in $M$) such that each component $L_i$ of $L$ is given a framing, i.e., a homotopy class of trivializations of the normal bundle. Such a framing of $L_i$ may be given as a homotopy class of a simple closed curve $\gamma_i$ in the boundary $\partial N(L_i)$ of a tubular neighborhood $N(L_i)$ of $L_i$ in $M$ which is homotopic to $L_i$ in $N(L_i)$.

For a framed link $L \subset M$ as above, let $M_L$ denote the result from $M$ of surgery along $L$. This manifold is obtained from $M$ by removing the interiors of $N(L_i)$, and gluing a solid torus $D^2 \times S^1$ to $\partial N(L_i)$ so that the curve $\partial D^2 \times \{*\}$, $* \in S^1$, is attached to $\gamma_i \subset \partial N(L_i)$ for each $i = 1, \ldots, l$.

Surgery along a framed link can be defined by using 4–manifolds as well. Let $L$ be a framed link in $M$. Let $W_L$ denote the 4–manifold obtained from the cylinder $M \times I$ by attaching a 2–handle $h_i \cong D^2 \times D^2$ along $N(L_i) \times \{1\}$ using the a homeomorphism

$$S^1 \times D^2 \overset{\cong}{\to} N(L_i),$$

which maps $S^1 \times \{*\}$, $* \in \partial D^2$, onto the framing $\gamma_i$. We have a natural identification

$$\partial W_L \cong M \underset{\partial M}{\cup} (\partial M \times I) \underset{\partial M_L}{\cup} M_L,$$

Thus, $W_L$ is a cobordism between $M$ and $M_L$. Note that $\partial W_L$ is connected if $\partial M \neq \emptyset$.

We define two moves on framed links. A *handle-slide* replaces one component $L_i$ of $L$ with a band sum $L'_i$ of $L_i$ and a parallel copy of another component $L_j$ as in Figure 1, where the blackboard framing convention is used. A *stabilization* adds to or removes from a link $L$ an isolated $\pm 1$–framed unknot.

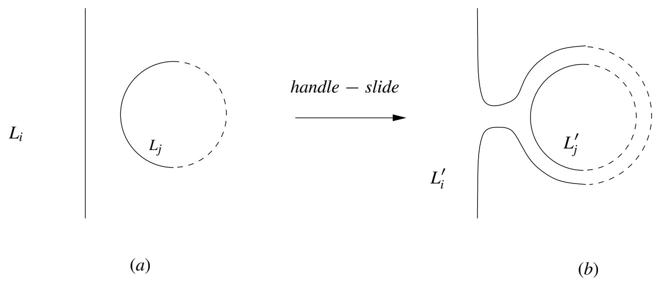



Figure 1: (a) Two components $L_i$ and $L_j$ of a framed link. (b) The result of a handle-slide of $L_i$ over $L_j$.

## 2.1 Some notations

We introduce some notations which we need in the statement of our generalization of Theorem 1.1, and which will be used in later sections as well.

Let $M$ be a compact, connected, oriented 3–manifold with *nonempty* boundary.

Let $F_1, \dots, F_n$ ($n \geq 1$) denote the components of $\partial M$. For each $k = 1, \dots, n$, choose a base point $p_k \in F_k$. We denote by $\pi_1(M; p_1, p_k)$ the set of homotopy classes of paths from $p_1$ to $p_k$ in $M$. We consider $p_1$ as the base point of $M$, and write

$$\pi_1(M) = \pi_1(M; p_1) = \pi_1(M; p_1, p_1).$$

Let $L$ be a framed link in $M$ as before. We consider the 4–manifold $W_L$ defined in Section 2. For $k = 1, \dots, n$, set $p_k^L = p_k \times \{1\} \in \partial M_L$ and $\gamma_k = p_k \times I \subset \partial W_L$. Note that $\gamma_k$ is an arc in $\partial W$ from $p_k \in \partial M \subset \partial W_L$ to $p_k^L$.

The inclusions

$$M \overset{i}{\hookrightarrow} W_L \overset{i'}{\hookleftarrow} M_L$$

induce surjective maps

$$\pi_1(M; p_1, p_k) \xrightarrow{i_k} \pi_1(W_L; p_1, p_k) \xleftarrow{i'_k} \pi_1(M_L; p_1^L, p_k^L)$$

for $k = 1, \dots, n$. Here $i'_k$ is defined to be the composition

$$\pi_1(M_L; p_1^L, p_k^L) \xrightarrow{i'_k} \pi_1(W_L; p_1^L, p_k^L) \underset{\gamma_1, \gamma_k}{\cong} \pi_1(W_L; p_1, p_k),$$

where the second isomorphism is induced by the arcs $\gamma_1$ and $\gamma_k$.

We regard $p_1^L$ as the base point of $M_L$ and write $\pi_1(M_L) := \pi_1(M_L; p_1^L)$. We regard $p_1$ as a base point of $W_L$ as well as of $M$, and we set $\pi_1(W_L) := \pi_1(W_L; p_1)$.

An Eilenberg–Mac Lane space $K(\pi_1(W_L), 1)$ can be obtained from $W_L$ by attaching cells which kill higher homotopy groups. Thus, there is a natural inclusion

$$\rho_L\colon W_L \hookrightarrow K(\pi_1(W_L), 1).$$

## 2.2 Construction of a homology class

Now, consider two framed links $L$ and $L'$ in $M$, and suppose that there exists a homeomorphism $h\colon M_L \to M_{L'}$ relative to the boundary. Moreover, we assume that there exist isomorphisms $f_k\colon \pi_1(W_L; p_1, p_k) \to \pi_1(W_{L'}; p_1, p_k)$ such that the diagram

(2)
$$\begin{tikzcd}
\pi_1(M_L; p_1^L, p_k^L) \arrow[rr, "h_k"] \arrow[d, "i'_k"'] & & \pi_1(M_{L'}; p_1^{L'}, p_k^{L'}) \arrow[d, "i'_k"] \\
\pi_1(W_L; p_1, p_k) \arrow[rr, "f_k"] & & \pi_1(W_{L'}; p_1, p_k) \\
& \pi_1(M; p_1, p_k) \arrow[ul, "i_k"] \arrow[ur, "i_k"'] &
\end{tikzcd}$$

commutes for $k = 1, \ldots, n$. For $k = 2, \ldots, n$, that $f_k$ *is an isomorphism* means that $f_k$ is a bijection. (Here, if $f_k$ is a bijection which makes the above diagram commutes, then it follows that $f_k$ is an isomorphism between the $\pi_1(W_L)$–set $\pi_1(W_L; p_1, p_k)$ and the $\pi_1(W_{L'})$–set $\pi_1(W_{L'}; p_1, p_k)$ along the group isomorphism $f_1\colon \pi_1(W_L) \to \pi_1(W_{L'})$.)

In the following, we define a homology class

$$\rho_*([W]) \in H_4(\pi_1(W_L), \mathbb{Z}),$$

by constructing a closed 4–manifold $W$ and a map $\rho\colon W \to K(\pi_1(W_L), 1)$.

As in [6], define a 4–manifold $W$ by

$$W := W_L \cup_\partial (-W_{L'}),$$

where we glue $W_L$ and $-W_{L'}$ (the orientation reversal of $W_{L'}$) along the boundaries using the identity map on $M \cup (\partial M \times I)$ and the homeomorphism $h\colon M_L \xrightarrow{\cong} M_{L'}$.

Consider the following diagram

(3)
$$\begin{tikzcd}
\partial W_L \arrow[r, "u'"] \arrow[d, "u"'] & W_{L'} \arrow[d, "j'"'] \arrow[ddr, "\tilde{\rho}_{L'}"] & \\
W_L \arrow[r, "j"] \arrow[drr, "\rho_L"'] & W \arrow[dr, "\rho"] & \\
& & K(\pi_1(W_L), 1),
\end{tikzcd}$$

where $u, u', j, j'$ are inclusions. The map $\tilde{\rho}_{L'}\colon W_{L'} \to K(\pi_1(W_L), 1)$ is the composite

$$W_{L'} \overset{\rho_{L'}}{\to} K(\pi_1(W_{L'}), 1) \underset{\simeq}{\overset{K(f_1^{-1},1)}{\to}} K(\pi_1(W_L), 1).$$

Here $K(f_1^{-1}, 1)$ is a homotopy equivalence, unique up to homotopy. By the definition of $W$, the square is a pushout. Hence, to prove existence of $\rho$ such that $\rho j = \rho_L$ and $\rho j' = \tilde{\rho}_{L'}$, we need only to show that $\rho_L u \simeq \tilde{\rho}_{L'} u'$, which easily follows from Lemma 2.1 below. (Proof of this lemma is the place where commutativity of (2) in Theorem 2.2 is necessary not only for $k = 1$ but also for $k = 2, \ldots, n$.)

**Lemma 2.1** *Under the above situation, the following diagram commutes.*

(4)
$$\begin{tikzcd}
\pi_1(\partial W_L) \arrow[r, "u'_*"] \arrow[d, "u_*"'] & \pi_1(W_{L'}) \arrow[d, "j'_*"] \\
\pi_1(W_L) \arrow[r, "j_*"'] \arrow[ur, "f_1"] & \pi_1(W)
\end{tikzcd}$$

**Proof** Since $u_*$ is surjective and the square is commutative, $u'_* = f_1 u_*$ implies $j_* = j'_* f_1$.

Let us prove $u'_* = f_1 u_*$. For $k = 2, \ldots, n$, choose an arc $c_k$ in $M$ from $p_1$ to $p_k$ disjoint from $L$. Set

$$d_k = (c_k \times \{0, 1\}) \cup (\partial c_k \times I),$$

which is a loop in $\partial W_L$ based at $p_1$. The fundamental group $\pi_1(\partial W_L)$ is then generated by the elements $d_2, \ldots, d_n$ and the images of the maps $i_*\colon \pi_1(M) \to \pi_1(\partial W_L)$ and $i'_*\colon \pi_1(M_L) \to \pi_1(\partial W_L)$. Hence $u'_* = f_1 u_*$ is reduced to the following:

(a) $u'_* i_* = f_1 u_* i_*\colon \pi_1(M) \to \pi_1(W_{L'})$,

(b) $u'_* i'_* = f_1 u_* i'_*\colon \pi_1(M_L) \to \pi_1(W_{L'})$,

(c) $u'_*(d_k) = f_1 u_*(d_k)$ for $k = 2, \ldots, n$.

(a) (resp. (b)) follows from commutativity of the lower (resp. upper) part of Diagram (2) for $k = 1$. (c) follows from commutativity of Diagram (2) for $k = 2, \ldots, n$. □

## 2.3 Statement of the theorem

Now we can state our generalization of Theorem 1.1 to 3–manifolds with boundary.

**Theorem 2.2** *Let $M$ be a compact, connected, oriented 3–manifold with $n > 0$ boundary components, and let $L, L' \subset M$ be framed links. Then the following conditions are equivalent.*

*(i) $L$ and $L'$ are related by a sequence of stabilizations and handle-slides.*

*(ii) There exist a homeomorphism $h\colon M_L \to M_{L'}$ relative to the boundary and isomorphisms $f_k\colon \pi_1(W_L; p_1, p_k) \to \pi_1(W_{L'}; p_1, p_k)$ for $k = 1, \ldots, n$ such that diagram (2) commutes for $k = 1, \ldots, n$ and $\rho_*([W]) = 0 \in H_4(\pi_1(W_L))$.*

**Remark 2.3** Theorem 1.1 can be derived from the case $\partial M = S^2$ of Theorem 2.2.

**Remark 2.4** In a paper in preparation [9], we will give an example in which a nonzero homology class $\rho_*([W])$ is realized.

## 2.4 Proof of the theorem

We need the following lemma which gives a necessary and sufficient condition for $\rho_*([W]) \in H_4(\pi_1(W_L))$ to vanish.

**Lemma 2.5** ([5, Lemma 9], [6, Lemma 2.1]) *In the situation of Theorem 2.2, we have $\rho_*([W]) = 0$ if and only if the connected sum of $W$ with some copies of $\pm\mathbb{C}P^2$ is the boundary of an oriented 5–manifold $\Omega$ in such a way that the diagram*

$$\begin{tikzcd} \pi_1(W_L; p_1) \arrow[rr, "f_1"] \arrow[dr, "j_*"'] & & \pi_1(W_{L'}; p_1) \arrow[dl, "j'_*"] \\ & \pi_1(\Omega; p_1) & \end{tikzcd} \tag{5}$$

*commutes and $j_*, j'_*$ are split injections induced by the inclusions $j\colon W_L \hookrightarrow \Omega$ and $j'\colon W_{L'} \hookrightarrow \Omega$.*

**Proof of Theorem 2.2** The proof that (i) implies (ii) is almost the same as the proof of Theorem 1.1 given in [5]. It follows from the "if" part of Lemma 2.5 and the fact that handle-slides and stabilizations on a framed link $L$ preserve the homeomorphism class of $M_L$ and the $\pi_1(W_L; p_1, p_k)$, $k = 1, \ldots, n$.

Now we prove that (ii) implies (i). Assume that all the algebraic conditions are satisfied. By Lemma 2.5, we may assume, after some stabilizations, that $W = \partial\Omega$, where $\Omega$ is a 5–manifold such that Diagram (5) commutes and $j_*$ and $j'_*$ are split injections. Now we alter $\Omega$, as in the original proof in [5], by doing surgery on $\Omega$ until we have

$\pi_1(\Omega) \cong \pi_1(W_L)$. Then we modify $L$ and $L'$ to $\tilde{L}$ and $\tilde{L}'$ by some specific stabilizations and handle-slides until we obtain a trivial cobordism $\Omega'$ joining $W_{\tilde{L}}$ and $W_{\tilde{L}'}$. Thus $W_{\tilde{L}}$ and $W_{\tilde{L}'}$ are two different relative handle decompositions of the same manifold.

By a famous theorem of J. Cerf [2] any two relative handle decomposition of the same manifold are connected by a sequence of handle slides, creating/ annihilating canceling handle pairs and isotopies. (For a reference see [7, Theorem 4.2.12].) Note that Cerf's theorem applies in the case when $W_L$ has two boundary components, as well as in the case where the boundary of the 4–manifold is connected. Fenn and Rourke have shown in [5] that these handle slides (1-handle slides and 2-handle slides) and creating or annihilating canceling handle pairs can be achieved by modifying the links using stabilization and handle-slides. Hence the proof is complete. □

## 3 Null-homotopic framed links

In this section we apply Theorem 2.2 to null-homotopic framed links.

Let $M$ be a compact, connected, oriented 3–manifold with $n > 0$ boundary components as before. We use the notations given in Section 2.

A framed link $L$ in $M$ is said to be *null-homotopic* if each component of $L$ is null-homotopic in $M$. In this case, the map

$$i_k\colon \pi_1(M; p_1, p_k) \to \pi_1(W_L; p_1, p_k)$$

is bijective for $k = 1, \ldots, n$. Define

$$e_k\colon \pi_1(M_L; p_1^L, p_k^L) \to \pi_1(M; p_1, p_k)$$

to be the composition

$$e_k\colon \pi_1(M_L; p_1^L, p_k^L) \xrightarrow{i'_k} \pi_1(W_L; p_1, p_k) \underset{\cong}{\xrightarrow{i_k^{-1}}} \pi_1(M; p_1, p_k),$$

which is surjective.

**Theorem 3.1** *Let $M$ be a compact, connected, oriented 3–manifold with $n > 0$ boundary components, and let $L, L' \subset M$ be null-homotopic framed links. Then the following conditions are equivalent.*

*(i) $L$ and $L'$ are related by a sequence of stabilizations and handle-slides.*

(ii) *There exists a homeomorphism $h\colon M_L \to M_{L'}$ relative to the boundary such that the following diagram commutes for $k = 1, \dots, n$.*

$$
\text{(6)} \qquad
\begin{tikzcd}
\pi_1(M_L; p_1^L, p_k^L) \arrow[rr, "h_k"] \arrow[dr, "e_k"'] & & \pi_1(M_{L'}; p_1^{L'}, p_k^{L'}) \arrow[dl, "e'_k"] \\
& \pi_1(M; p_1, p_k) &
\end{tikzcd}
$$

**Remark 3.2** For a closed 3–manifold $M$, the variant of Theorem 3.1 is implicitly obtained in [5]. Two null-homotopic framed links $L$ and $L'$ in a closed, connected, oriented 3–manifold $M$ are related by a sequence of stabilizations and handle-slides if and only if there is a homeomorphism $h\colon M_L \to M_{L'}$ such that the diagram

$$
\text{(7)} \qquad
\begin{tikzcd}
\pi_1(M_L) \arrow[rr, "h_*"] \arrow[dr, "e"'] & & \pi_1(M_{L'}) \arrow[dl, "e'"] \\
& \pi_1(M) &
\end{tikzcd}
$$

commutes. Here $e$ and $e'$ are defined similarly as before.

Theorem 3.1 follows easily from Theorem 2.2 and the following lemma, which seems to be well known. In fact, it seems implicit in Fenn and Rourke [5], p. 8, ll. 8–9, where it reads "For many other groups, $\eta(\Delta)$ vanishes, e.g. the fundamental group of any 3–manifold." We give a sketch of proof of this fact since we have not been able to find a suitable reference.

**Lemma 3.3** *If $M$ is a compact, connected, oriented 3–manifold, then we have $H_4(\pi_1 M, \mathbb{Z}) = 0$.*

**Proof** Consider a connected sum decomposition $M \cong M_1 \sharp \dots \sharp M_k$, $k \geq 0$, where each $M_i$ is prime. Since $\pi_1 M \cong \pi_1 M_1 * \cdots * \pi_1 M_k$, we have

$$H_4(\pi_1 M, \mathbb{Z}) \cong H_4(\pi_1 M_1, \mathbb{Z}) \oplus \cdots \oplus H_4(\pi_1 M_k, \mathbb{Z}).$$

Thus, we may assume without loss of generality that $M$ is prime. If $M = S^2 \times S^1$, then we have $H_4(\pi_1 M, \mathbb{Z}) = H_4(\mathbb{Z}, \mathbb{Z}) = 0$. Hence we may assume that $M$ is irreducible.

If $\pi_1 M$ is infinite, then $M$ is a $K(\pi_1 M, 1)$ space. Hence

$$H_4(\pi_1 M, \mathbb{Z}) \cong H_4(M, \mathbb{Z}) = 0.$$

Suppose that $\pi_1 M$ is finite. If $\partial M \neq \emptyset$, then we have $M \cong B^3$ and clearly $H_4(\pi_1 M, \mathbb{Z}) = 0$. Thus we may assume that $M$ is closed. Then the universal cover of $M$ is a homotopy 3–sphere, which is $S^3$ by the Poincaré conjecture established by Perelman. By Lemma 6.2 of [1], we have

$$H^5(\pi_1 M, \mathbb{Z}) \cong H^1(\pi_1 M, \mathbb{Z}). \tag{8}$$

Recall that, for any finite group $G$, $H_n(G, \mathbb{Z})$ is finite for all $n \geq 1$. This fact and the universal coefficient theorem imply

$$H^1(\pi_1 M, \mathbb{Z}) \cong \mathrm{Hom}(H_1(\pi_1 M, \mathbb{Z}), \mathbb{Z}) = 0, \tag{9}$$

$$H^5(\pi_1 M, \mathbb{Z}) \cong \mathrm{Hom}(H_5(\pi_1 M, \mathbb{Z}), \mathbb{Z}) \oplus \mathrm{Ext}(H_4(\pi_1 M, \mathbb{Z}), \mathbb{Z}) \cong H_4(\pi_1 M, \mathbb{Z}), \tag{10}$$

where the last $\cong$ follows since $H_4(\pi_1 M, \mathbb{Z})$ is finite. Now, (8), (9) and (10) imply that $H_4(\pi_1 M, \mathbb{Z}) = 0$. □

# 4 $\pi_1$–admissible framed links

In this section we consider $\pi_1$–admissible framed links and give a refinement of Theorem 3.1. We also consider $\pi_1$–admissible framed links in cylinders over surfaces.

## 4.1 $\pi_1$–admissible framed links in 3–manifolds

Let $M$ be a compact, connected, oriented 3–manifold. Let us call a framed link $L$ in $M$ *$\pi_1$–admissible* if

- $L$ is null-homotopic, and
- the linking matrix of $L$ is diagonal with diagonal entries $\pm 1$, or, in other words, $L$ is algebraically split and $\pm 1$–framed.

Surgery along $\pi_1$–admissible framed links has been studied by Cochran, Gerges and Orr [3]. (They considered mainly more general framed links.) They proved that for all $d \geq 1$, we have $\pi_1(M_L)/\Gamma_d \pi_1(M_L) \cong \pi_1(M)/\Gamma_d \pi_1(M)$, where for a group $G$, $\Gamma_d G$ denotes the $d$th lower central series subgroup of $G$ defined by $\Gamma_1 G = G$ and $\Gamma_d G = [G, \Gamma_{d-1} G]$ for $d \geq 2$. In this sense, surgery along a $\pi_1$–admissible framed link $L$ in a 3–manifold $M$ gives a 3–manifold $M_L$ whose fundamental group is very close to that of $M$.

Surgery along $\pi_1$–admissible framed links was also studied by the first author [8]. To state the result from [8] that we use in this section, we introduce "band-slides" and "Hoste moves", which are two special kinds of moves on $\pi_1$–admissible framed links.

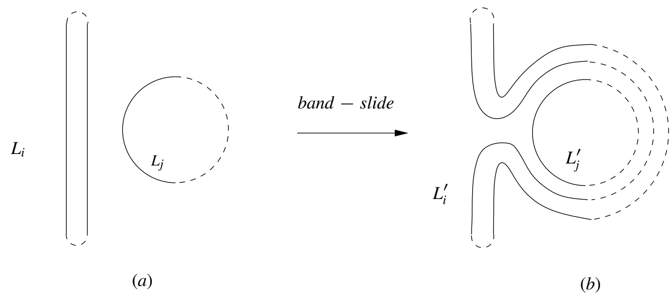



Figure 2: (a) Two components $L_i$ and $L_j$ of a framed link. (b) The result of a band-slide of $L_i$ over $L_j$.

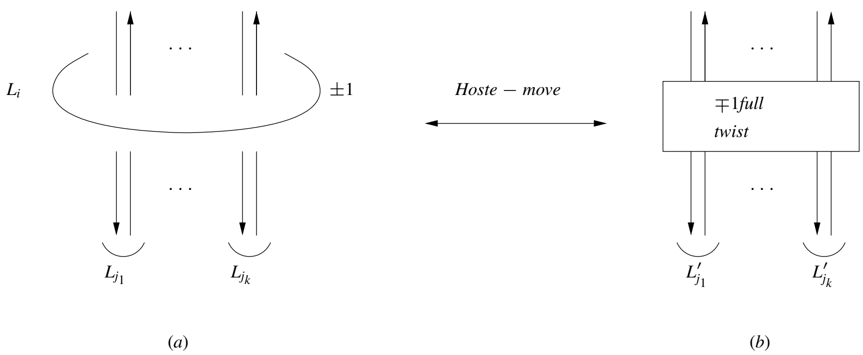



Figure 3: (a) The component $L_i$ of $L$ is unknotted and of framing $\pm 1$. (b) The result $L'_{L_i}$ of a Hoste move on $L_i$.

A *band-slide* is a pair of algebraically cancelling pair of handle-slides of one component over another, see Figure 2. A band-slide on a $\pi_1$–admissible framed link produces a $\pi_1$–admissible framed link.

A *Hoste move* is depicted in Figure 3. Let $L = L_1 \cup \cdots \cup L_l$ be a $\pi_1$–admissible framed link in $M$, with an unknotted component $L_i$ with framing $\epsilon = \pm 1$. Since $L$ is $\pi_1$–admissible, the linking number of $L_i$ and each component of $L' := L \setminus L_i$ is zero. Let $L'_{L_i}$ denote the framed link obtained from $L'$ by surgery along $L_i$, which is regarded as a framed link in $M \cong M_{L_i}$. The link $L'_{L_i}$ is again $\pi_1$–admissible. Then the framed links $L$ and $L'_{L_i}$ are said to be related by a Hoste move.

**Proposition 4.1** ([8, Proposition 6.1]) *For two $\pi_1$–admissible framed links $L$ and $L'$ in a connected, oriented 3–manifold $M$, the following conditions are equivalent.*

*(i)* *$L$ and $L'$ are related by a sequence of stabilizations and handle-slides.*

*(ii)* *$L$ and $L'$ are related by a sequence of stabilizations and band-slides.*

*(iii)* *$L$ and $L'$ are related by a sequence of Hoste moves.*

Theorem 3.1 and Proposition 4.1 immediately imply the following result.

**Theorem 4.2** *Let $M$ be a compact, connected, oriented 3–manifold with $n > 0$ boundary components, and let $L, L' \subset M$ be $\pi_1$–admissible, framed links. Then the following conditions are equivalent.*

*(i)* *$L$ and $L'$ are related by a sequence of stabilizations and band-slides.*

*(ii)* *$L$ and $L'$ are related by a sequence of Hoste moves.*

*(iii)* *There exists a homeomorphism $h\colon M_L \to M_{L'}$ relative to the boundary such that the following diagram commutes for $k = 1, \ldots, n$.*

(11)
$$\begin{tikzcd} \pi_1(M_L; p_1^L, p_k^L) \arrow[rr, "h_k"] \arrow[dr, "e_k"'] & & \pi_1(M_{L'}; p_1^{L'}, p_k^{L'}) \arrow[dl, "e'_k"] \\ & \pi_1(M; p_1, p_k) & \end{tikzcd}$$

## 4.2 $\pi_1$–admissible framed links in cylinders over surfaces

In this subsection, we consider the special cases of Theorem 4.2 where $M = \Sigma_{g,n} \times I$ is the cylinder over a surface $\Sigma_{g,n}$ of genus $g \geq 0$ with $n \geq 0$ boundary components. In this case, the condition (3) in Theorem 4.2 can be weakened.

Let $L$ be a $\pi_1$–admissible framed link in the cylinder $M = \Sigma_{g,n} \times I$. By [3, Theorem 6.1], there are natural isomorphisms between nilpotent quotients

(12)
$$\pi_1 M_L / \Gamma_d \pi_1 M_L \cong \pi_1 M / \Gamma_d \pi_1 M \cong \pi_1 \Sigma_{g,n} / \Gamma_d \pi_1 \Sigma_{g,n}.$$

for all $d \geq 1$.

### 4.2.1 Surfaces with nonempty boundary

Consider the case $n \geq 1$. Note that $\partial M = \partial(\Sigma_{g,n} \times I)$ is connected.

**Proposition 4.3** *Let $L$ and $L'$ be two $\pi_1$–admissible, framed links in $M = \Sigma_{g,n} \times I$ with $n > 0$. Then the following conditions are equivalent.*

*(i)* *$L$ and $L'$ are related by a sequence of stabilizations and band-slides.*

*(ii)* *There exists a homeomorphism $h\colon M_L \to M_{L'}$ relative to the boundary.*

**Proof** That (i) implies (ii) immediately follows from Theorem 4.2.

To prove (ii) implies (i), one has to show that the diagram (11) commutes for $k = 1$, i.e.,

(13)
$$\begin{tikzcd} \pi_1(M_L; p_1^L) \arrow[rr, "h_1"] \arrow[dr, "e_1"'] & & \pi_1(M_{L'}; p_1^{L'}) \arrow[dl, "e'_1"] \\ & \pi_1(M; p_1) & \end{tikzcd}$$

commutes. This can be checked by using the isomorphism (12). Let $x \in \pi_1(M_L; p_1^L)$. For $d \geq 1$, take the nilpotent quotient of Diagram (13)

(14)
$$\begin{tikzcd} \pi_1(M_L; p_1^L)/\Gamma_d \arrow[rr, "h_1", "\cong"'] \arrow[dr, "\cong", "e_1"'] & & \pi_1(M_{L'}; p_1^{L'})/\Gamma_d \arrow[dl, "\cong"', "e'_1"] \\ & \pi_1(M; p_1)/\Gamma_d & \end{tikzcd}$$

where all arrows are isomorphisms. Since the homeomorphism $h\colon M_L \xrightarrow{\cong} M_{L'}$ respects the boundary, Diagram (14) commutes. Hence, for $x \in \pi_1(M_L; p_1^L)$ we have

(15)
$$e_1(x) \equiv e'_1 h_1(x) \pmod{\Gamma_d \pi_1(M; p_1)}.$$

Since (15) holds for all $d \geq 1$, and since we have $\bigcap_{d\geq 1} \Gamma_d \pi_1(M; p_1) = \{1\}$, it follows that $e_1(x) = e'_1 h_1(x)$. Hence Diagram (13) commutes. □

### 4.2.2 Closed surfaces

Now, we consider the case $n = 0$. In this case, the manifold $M = \Sigma_{g,0} \times I$ has two boundary components. Set $F_1 = \Sigma_{g,0} \times \{0\}$ and $F_2 = \Sigma_{g,0} \times \{1\}$. Choose a base point $p$ of $\Sigma_{g,0}$ and set $p_1 = (p, 0) \in F_1$ and $p_2 = (p, 1) \in F_2$.

**Proposition 4.4** *Let $L$ and $L'$ be two $\pi_1$–admissible, framed links in $M = \Sigma_{g,0} \times I$. Then the following conditions are equivalent:*

*(i)* *$L$ and $L'$ are related by a sequence of stabilizations and band-slides,*

(ii) *There exists a homeomorphism $h\colon M_L \to M_{L'}$ relative to the boundary such that the following diagram commutes:*

$$\begin{tikzcd} \pi_1(M_L; p_1^L, p_2^L) \arrow[rr, "h_2"] \arrow[dr, "e_2"'] & & \pi_1(M_{L'}; p_1^{L'}, p_2^{L'}) \arrow[dl, "e_2'"] \\ & \pi_1(M; p_1, p_2) & \end{tikzcd} \tag{16}$$

**Proof** The proof is similar to that of (ii) implies (i) for Proposition 4.3; one has to prove that the diagram (11) commutes for $k = 1$. This can be done similarly using the fact that

$$\bigcap_{d\geq 1} \Gamma_d \pi_1(M; p_1) = \bigcap_{d\geq 1} \Gamma_d \pi_1(\Sigma_{g,0}; p_1) = \{1\}.$$

□

For the cylinder over the torus $T^2 = \Sigma_{1,0}$, we do not need commutativity of (16) in Proposition 4.4.

**Proposition 4.5** *Let $L$ and $L'$ be two $\pi_1$–admissible, framed links in the cylinder $M = T^2 \times I$. Then the following conditions are equivalent.*

(i) *$L$ and $L'$ are related by a sequence of stabilizations and band-slides.*

(ii) *There exists a homeomorphism $h\colon M_L \to M_{L'}$ relative to the boundary.*

**Proof** By Proposition 4.4 we just have to show that if there exists a homeomorphism $h\colon M_L \to M_{L'}$ relative to the boundary, then there exists a homeomorphism $h'\colon M_L \to M_{L'}$ such that the diagram (16), with $h_2$ replaced by $h_2'$, commutes.

Consider the cylinder $T^2 \times I$. Fix one boundary component while twisting the other once along the meridian (resp. the longitude) of $T^2$. This defines a self-homeomorphism $\tau_m$ (resp. $\tau_l$) on $T^2 \times I$ relative to the boundary which maps $\{*\} \times I$, $* \in T^2$, to a line with the same endpoints but which travels once along the meridian (resp. the longitude). A sequence of $\tau_m$ and $\tau_l$ defines a self-homeomorphism $s$ on $T^2 \times I$ by using the composition of maps. Any bijective map $b\colon \pi_1(T^2 \times I; p_1, p_2) \to \pi_1(T^2 \times I; p_1, p_2)$ of $\pi_1(T^2 \times I)$–sets can be induced by such a self-homeomorphism. Let

$$M'_{L'} = M_{L'} \cup_{T^2} (T^2 \times I)$$

be a homeomorphic copy of $M_{L'}$ obtained by gluing together $M_{L'}$ and $T^2 \times I$ along $F_2 \cong T^2 \subset M_{L'}$ and $T^2 \times \{0\}$ using the identity map. Any self-homeomorphism $s$ on

$T^2 \times I$ as defined above, extends to a self-homeomorphism $\tilde{s}$ on $M'_{L'}$. Thus, we can find a self-homeomorphism $s$ on $T^2 \times I$ such that the composition $h' = \tilde{s} \circ h$ defines a commutative diagram (16).

□

**Remark 4.6** If $g > 1$, then the above proof can not be extended to the closed surface $\Sigma_{g,0}$. In this case, every self-homeomorphism of $\Sigma_{g,0}$ is homotopic to the identity. This can be seen as follows. Every diffeomorphism $g \in \mathrm{Diff}(\Sigma_{g,0} \times I)$ relative to the boundary is homotopic to a diffeomorphism $g'(x,t) := (g_t(x), t)$ with $g_t(x) \in \mathrm{Diff}(\Sigma_{g,0})$. Since $g$ is the identity on the boundaries we have $g_0(x) = g_1(x) = \mathrm{id}_{\Sigma_{g,0}}(x)$. Hence, $g_t$ defines a loop in $\mathrm{Diff}(\Sigma_{g,0})$ and every $g_t$ is homotopic to $\mathrm{id}_{\Sigma_{g,0}}$. Thus, $g_t$ is a loop in the group $\mathrm{Diff}_0(\Sigma_{g,0})$ of diffeomorphisms of $\Sigma_{g,0}$ homotopic to the identity. By a theorem of Earle and Eells [4] the group $\mathrm{Diff}_0(\Sigma_{g,0})$ is contractible when $g > 1$. Hence, the loop formed by $g_t$ is homotopic to $\mathrm{id}_{\Sigma_{g,0}}$ and therefore $g$ is homotopic to $\mathrm{id}_{\Sigma_{g,0}\times I}$.

# 5 Example

## 5.1 An example

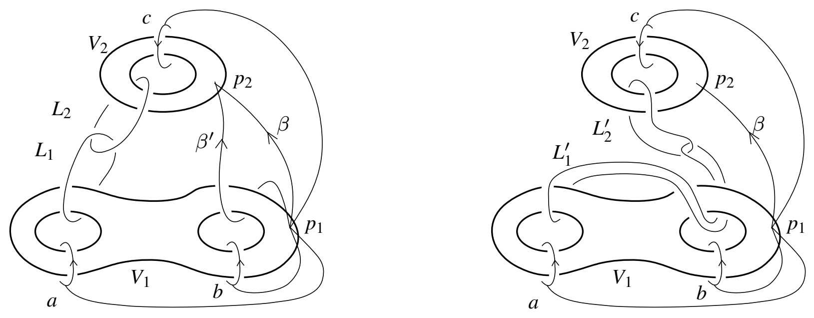
Let us call the equivalence relation on framed links generated by stabilizations and handle-slides the *$\delta$–equivalence*.

The following example shows that commutativity of diagram (2) for $k = 2, \dots, n$ is necessary as well as that for $k = 1$.

Let $V_1$ and $V_2$ be handlebodies of genus 2 and 1, respectively, embedded in $S^3$ in a trivial way, and set $M = S^3 \setminus \mathrm{int}(V_1 \cup V_2)$, $F_k = \partial V_k$ $(k = 1, 2)$, see Figure 4(a). Let $\beta, \beta' \subset M$ be two arcs from $p_1 \in F_1$ to $p_2 \in F_2$, and let $a, b$ and $c$ be loops based at $p_1$, as depicted. The fundamental group $\pi_1 M$ is freely generated by $a, b, c \in \pi_1 M$.

Let $L = L_1 \cup L_2$ be the framed link in $M$ as depicted in Figure 4 (a), where $L_1$ and $L_2$ are of framing $0$. The result $M_L$ of surgery along $L$ is obtained from $M$ by letting the two handles in $V_1$ and $V_2$ clasp each other. $\pi_1 M_L$ has a presentation $\langle a, b, c \mid aca^{-1}c^{-1} = 1 \rangle$.

Let $f \colon M \xrightarrow{\cong} M$ be a homeomorphism relative to the boundary such that $f(\beta') = \beta$. The image $f(L) = L' = L'_1 \cup L'_2$ looks as depicted in Figure 4(b). Let $h \colon M_L \xrightarrow{\cong} M_{L'}$ be




(a) (b)

Figure 4

the homeomorphism induced by $f$. Note that $\pi_1 W_L \cong \langle b \rangle \cong \mathbb{Z}$ and $\pi_1 W_{L'} \cong \langle b \rangle \cong \mathbb{Z}$. Observe that diagram (2) is commutative for $k = 1$ but not for $k = 2$. Hence Theorem 2.2 can not be used here to deduce that $L$ and $L'$ are $\delta$–equivalent.

In fact, $L$ and $L'$ are *not* $\delta$–equivalent. We can verify this fact as follows. Let $T$ be a tubular neighborhood of $\beta$ in $M$. Let $K$ be a small 0–framed unknot meridional to $T$. Let $J$ be a knot in $\operatorname{int} V_1$, to which the loop $b$ is meridional, as depicted in Figure 5(a), (b), and let $N(J)$ denote a small tubular neighborhood of $J$ in $V_1$. Set $M' = S^3 \setminus \operatorname{int} N(J)$, which is homeomorphic to a solid torus. Let $K_1$ and $K_2$ be framed knots as depicted. It suffices to prove that the framed links $\tilde{L} = L \cup K \cup K_1 \cup K_2$ and $\tilde{L}' = L' \cup K \cup K_1 \cup K_2$ in $M'$ are not $\delta$–equivalent. Observe that $\tilde{L}$ (resp. $\tilde{L}'$) is $\delta$–equivalent to the 3–component link depicted in Figure 5(c) (resp. (d)). (These links are the Borromean rings in $S^3$ with 0–framings.) One can show that these two links are not $\delta$–equivalent by using the invariant $B$ of framed links defined in Subsection 5.2 below. For the framed links $L_c$ and $L_d$ of Figure 5 (c) and (d), respectively, we have $B(L_c) = \{0\}$ and $B(L_d) = \mathbb{Z}$.

## 5.2 An invariant of $O_n$–$\pi_1$–admissible framed links in the exterior of an unknot in $S^3$

For $n \geq 0$, let $O_n$ and $I_n$ denote the zero matrix and the identity matrix, respectively, of size $n$. For $p, q \geq 0$, set $I_{p,q} = I_p \oplus (-I_q)$, where $\oplus$ denotes block sum.

Let $J$ be an unknot in $S^3$ and set $E = S^3 \setminus \operatorname{int} N(J) \cong S^1 \times D^2$, where $N(J)$ is a tubular neighborhood of $J$.

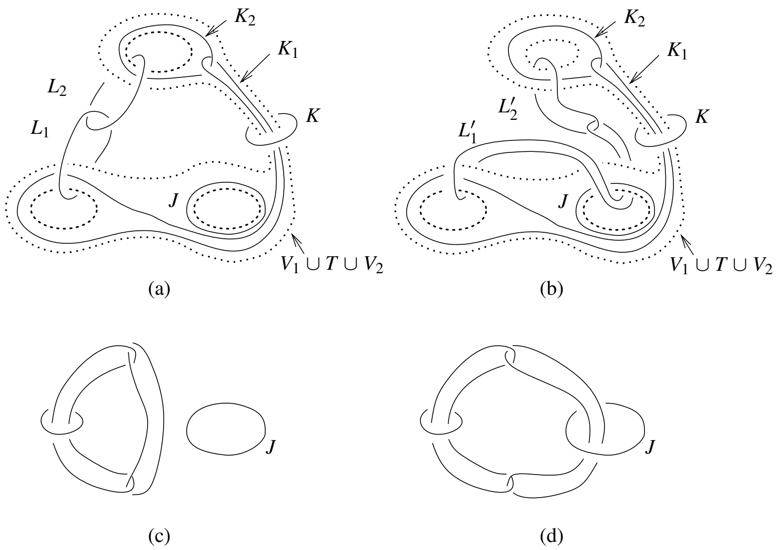



Figure 5

Let $L = L^z_1 \cup \cdots \cup L^z_n \cup L^a_1 \cup \cdots \cup L^a_{p+q}$, $n, p, q \geq 0$, be an oriented, ordered, null-homotopic framed link in $E$ whose linking matrix is of the form $O_n \oplus I_{p,q}$. Let us call such a framed link $O_n$–$\pi_1$–*admissible*. Let us call $L^z_1, \ldots, L^z_n$ the *z–components* of $L$, and $L^a_1, \ldots, L^a_{p+q}$ the *a–components* of $L$.

Since $L^z_1 \cup \cdots \cup L^z_n \cup J$ is algebraically split, for $1 \leq i < j \leq n$ the triple Milnor invariant $\bar{\mu}(L^z_i, L^z_j, J) \in \mathbb{Z}$ is well defined. Set

$$B(L) = \mathrm{Span}_{\mathbb{Z}}\{\bar{\mu}(L^z_i, L^z_j, J) \mid 1 \leq i < j \leq n\},$$

which is a subgroup of $\mathbb{Z}$. Note that $B(L)$ does not depend on the $a$–components of $L$. Note also that $B(L)$ does not depend on the ordering and orientations of the $z$–components of $L$.

**Lemma 5.1** *$B(L)$ is invariant under handle-slide of a $z$–component over another $z$–component.*

**Proof** It suffices to consider a handle-slide of $L^z_1$ over $L^z_2$. The link obtained from $L$ by this handle-slide is

$$L' = (L')^z_1 \cup (L')^z_2 \cup \cdots \cup (L')^z_n \cup (L')^a_1 \cup \cdots \cup (L')^a_{p+q},$$

where $(L')^z_1 = L^z_1 \sharp_b \tilde{L}^z_2$ is a band sum of $L^z_1$ and a parallel copy $\tilde{L}^z_2$ of $L^z_2$ along a band $b$, and $(L')^z_i = L^z_i$ for $i = 2, \ldots, n$. We have

$$\begin{aligned}
&\bar{\mu}((L')^z_1, (L')^z_2, J) = \bar{\mu}(L^z_1, L^z_2, J),\\
&\bar{\mu}((L')^z_1, (L')^z_i, J) = \bar{\mu}(L^z_1, L^z_i, J) + \bar{\mu}(L^z_2, L^z_i, J) \quad (2 \leq i \leq n),\\
&\bar{\mu}((L')^z_i, (L')^z_j, J) = \bar{\mu}(L^z_i, L^z_j, J) \quad (2 \leq i < j \leq n).
\end{aligned}$$

Hence we have $B(L') = B(L)$. □

**Lemma 5.2** *$B(L)$ is invariant under band-slides.*

**Proof** Clearly, a band-slide of an $a$–component over another ($z$– or $a$–) component preserves $B$. Lemma 5.1 implies that a band-slide of a $z$–component over another $z$–component preserves $B$.

Consider a band-slide of a $z$–component $L^z_1$ of $L$ over an $a$–component $L^a_1$ of $L$. Let $L'$ be the resulting link. Let $L''$ denote the result from $L$ by the same band-slide as before, but we use here the 0–framing of $L^a_1$ for the band-slide. By the previous case, it follows that $B(L'') = B(L)$. The $z$–part $(L')^z (= (L')^z_1 \cup \cdots \cup (L')^z_n)$ of $L'$ differs from the $z$–part $(L'')^z$ of $L''$ by self-crossing change of the component $(L')^z_1$. Since the triple Milnor invariant is invariant under link homotopy, it follows that $B(L') = B(L'')$. Hence $B(L) = B(L')$. □

**Proposition 5.3** *If two $O_n$–$\pi_1$–admissible framed links $L$ and $L'$ are $\delta$–equivalent, then we have $B(L) = B(L')$.*

**Proof** We give a sketch proof assuming familiarity with techniques on framed links developed in [8].

If $L$ and $L'$ are $\delta$–equivalent, then after adding to $L$ and $L'$ some unknotted $\pm 1$–framed components by stabilizations, $L$ and $L'$ become related by a sequence of handle-slides. Clearly, stabilization on an $O_n$–$\pi_1$–admissible framed link preserves $B$. So, we may assume that $L$ and $L'$ are related by a sequence of handle-slides. It follows that $L$ and $L'$ have the same linking matrix $O_n \oplus I_{p,q}$, $n, p, q \geq 0$.

Recall that for each sequence $S$ of handle-slides between oriented, ordered framed links there is an associated invertible matrix $\varphi(S)$ with coefficients in $\mathbb{Z}$, see e.g. [8]. In our case, a sequence from $L$ to $L'$ gives a matrix $P \in GL(n+p+q; \mathbb{Z})$ such that

$$P(O_n \oplus I_{p,q})P^t = (O_n \oplus I_{p,q}). \tag{17}$$

(Here $P^t$ denotes the transpose of $P$.) Let $H_{n,p,q} < GL(n{+}p{+}q; \mathbb{Z})$ denote the subgroup consisting of matrices satisfying (17). It is easy to see that $H_{n,p,q}$ is generated by the following elements.

(a) $Q \oplus I_{p+q}$, where $Q \in GL(n;\mathbb{Z})$.

(b) $\begin{pmatrix} I_n & 0 \\ X & I_{p+q} \end{pmatrix}$, where $X \in \mathrm{Mat}_{\mathbb{Z}}(p+q,n)$.

(c) $I_n \oplus R$, where $R \in O(p,q;\mathbb{Z}) = \{T \in GL(p+q;\mathbb{Z}) \mid TI_{p,q}T^t = I_{p,q}\}$.

Hence $\varphi(S)$ can be expressed as

$$\varphi(S) = w_1^{\epsilon_1} \cdots w_k^{\epsilon_k},$$

where $k \geq 0$, $\epsilon_1, \dots, \epsilon_k \in \{\pm 1\}$, and $w_1, \dots, w_k \in H_{n,p,q}$ are generators of the above form.

By an argument similar to that in [8], we can show that there are framed links $L^{(0)} = L, L^{(1)}, \dots L^{(k)} = L''$ such that

(i) for $i = 1, \dots, k$, $L^{(k-1)}$ and $L^{(k)}$ are related by a sequence $S_i$ of handle-slides, orientation changes and permutations with associated matrix $\varphi(S_i) = w_i^{\epsilon_i}$,

(ii) there is a sequence of band-slides from $L''$ and $L'$.

Here the framed links $L^{(0)}, \dots, L^{(k)}$ are $O_n$–$\pi_1$–admissible.

Let $i = 1, \dots, k$. If $w_i$ is a generator of type (b) or (c), then we have $B(L^{(i-1)}) = B(L^{(i)})$ since $S_i$ is a sequence of handle-slides of $a$–components over other ($z$– or $a$–) components. If $w_i$ is a generator of type (a), then $S_i$ is a sequence of orientation changes of $z$–components, permutations of $z$–components, and handle-slides of $z$–components over $z$–components. Clearly, orientation changes and permutations preserve $B$. Handle-slides of $z$–components over $z$–components also preserve $B$ by Lemma 5.1.

By Lemma 5.2, we have $B(L'') = B(L')$. Hence we have $B(L) = B(L')$. □

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# Chapter 6

# Kirby calculus for null-homologous framed links in $3$–manifolds

In this chapter we present a preprint of a joint work with K. Habiro.

# Kirby calculus for null-homologous framed links in 3–manifolds

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## Contents

# 1 Introduction

Surgery along a framed link $L$ in a 3–manifold $M$ is an operation which removes a tubular neighborhood of $L$ from $M$ and glues back a solid torus in a different way using the framing, and we obtain a new 3–manifold $M_L$. The 3–manifold $M_L$ can also be defined by using the 4–manifold $W_L$ obtained from the cylinder $M \times [0,1]$ by attaching 2-handles on $M \times \{1\}$ along $L \times \{1\}$. Then $W_L$ is a cobordism between $M_L$ and $M$.

Every closed, connected, oriented 3–manifold can be obtained from the 3-sphere $S^3$ by surgery along a framed link [15, 20]. Kirby's theorem [14] gives a criterion for two framed links in $S^3$ to produce orientation-preserving diffeomorphic result of surgery: Two framed links $L$ and $L'$ in $S^3$ yield orientation-preserving diffeomorphic 3–manifold if and only if $L$ and $L'$ are related by a sequence of two kinds of moves, called *stabilizations* and *handle-slides*, see Figure 1.

Fenn and Rourke [6] generalized Kirby's theorem to framed links in a general closed 3–manifold in two natural ways.

One the one hand, they proved that two framed links in $M$ yield orientation-preserving diffeomorphic 3–manifolds if and only if they are related by a sequence of stabilizations, handle-slides and $K_3$-*moves*. Here a $K_3$-move on a framed link adds or removes a 2-component sublink $K \cup K'$ such that $K$ is a framed knot in $M$ with arbitrary framing, and $K'$ is a small 0-framed knot

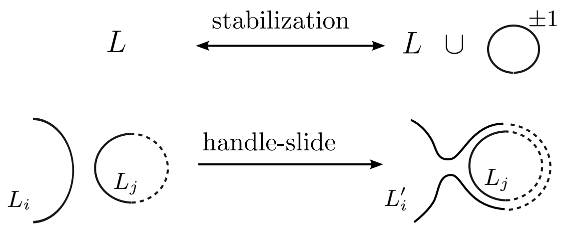



Figure 1: (a) A stabilization adds or deletes an isolated $\pm 1$-framed unknot. (b) A handle-slide replaces one component with the band-sum of the component with a parallel copy of another component.

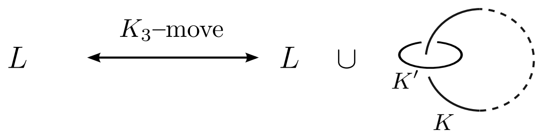



Figure 2: A $K_3$-move $L \leftrightarrow L \cup K \cup K'$.

meridional to $K$, see Figure 2. Roberts [18] generalized this result to 3–manifolds with boundary.

On the other hand, Fenn and Rourke considered the equivalence relation, called the *$\delta$-equivalence*, on framed links in $M$ generated by stabilizations and handle-slides. They proved that two framed links $L$ and $L'$ in a closed oriented 3–manifold $M$ are $\delta$-equivalent if and only if $\pi_1 W_L$ and $\pi_1 W_{L'}$ are isomorphic and there is an orientation-preserving diffeomorphism $h\colon M_L \to M'_L$ which satisfies a certain condition. This result is generalized to 3–manifolds with boundary, see [12]. (See also [7] for the case where the boundary is connected.)

The main purpose of this paper is to study calculus of null-homologous framed links in a compact, connected, oriented 3–manifold $M$ with non-empty boundary.

Let $k$ be $\mathbb{Z}$ or $\mathbb{Q}$. A framed link $L$ in $M$ is said to be *$k$-null-homologous* if every component of $L$ is $k$-null-homologous in $M$, i.e., represents $0 \in H_1(M;k)$.

Let $P \subset \partial M$ be a subset which contains exactly one point of each connected component of $\partial M$. To a $k$-null-homologous framed link $L$ in $M$ is associated a surjective homomorphism

$$g_L\colon H_1(M_L, P_L; k) \to H_1(M, P; k) \qquad (1.0.1)$$

defined as the composite

$$H_1(M_L, P_L; k) \xrightarrow{\mathrm{incl}_*} H_1(W_L, P; k) \underset{\cong}{\xrightarrow{\mathrm{incl}_*^{-1}}} H_1(M, P; k).$$

Here $P_L \subset \partial M_L$ is the image of $P$ by the natural identification map $\partial M \overset{\cong}{\to} \partial M_L$.

A *$k$-null-homologous $K_3$-move* is a $K_3$-move such that the component $K$ in the definition of a $K_3$-move is $k$-null-homologous. A $k$-null-homologous $K_3$-moves transforms a $k$-null-homologous framed link into another $k$-null-homologous framed link.

We will define an *IHX-move* in Section 6.3. This move corresponds to the IHX relation for tree claspers, closely related to the theory of finite type invariants of links and 3–manifolds.

The first main result in this paper is the following.

**Theorem 1.1.** *Let $M$ be a compact, connected, oriented 3–manifold with non-empty boundary. Let $P \subset \partial M$ be a subset containing exactly one point of each connected component of $\partial M$. Let $L$ and $L'$ be $\mathbb{Q}$-null-homologous framed links in $M$. Then the following conditions are equivalent.*

*(i) $L$ and $L'$ are related by a sequence of stabilization, handle-slides, $\mathbb{Q}$-null-homologous $K_3$-moves, and IHX-moves.*

*(ii) There is an orientation-preserving diffeomorphism $h\colon M_L \overset{\cong}{\to} M_{L'}$ restricting to the canonical identification $\partial M_L \cong \partial M_{L'}$ such that the following diagram commutes.*

$$\begin{tikzcd} H_1(M_L, P_L; \mathbb{Q}) \arrow[rr, "h_*", "\cong"'] \arrow[dr, "g_L"'] & & H_1(M_{L'}, P_{L'}; \mathbb{Q}) \arrow[dl, "g_{L'}"] \\ & H_1(M, P; \mathbb{Q}). & \end{tikzcd} \tag{1.0.2}$$

Theorem 1.1 is proved in Section 8.3 as Theorem 8.5.

If $H_1(M; \mathbb{Z})$ is free abelian, then $\mathbb{Q}$-null-homologous framed links in $M$ are $\mathbb{Z}$-null-homologous. It is easy to see that Theorem 1.1 implies the following.

**Theorem 1.2.** *Let $M$ and $P$ be as in Theorem 1.1, and assume that $H_1(M; \mathbb{Z})$ is free abelian. Let $L$ and $L'$ be $\mathbb{Z}$-null-homologous framed links in $M$. Then the following conditions are equivalent.*

*(i) $L$ and $L'$ are related by a sequence of stabilization, handle-slides, $\mathbb{Z}$-null-homologous $K_3$-moves, and IHX-moves.*

*(ii) There is an orientation-preserving diffeomorphism $h\colon M_L \longrightarrow M_{L'}$ restricting to the canonical identification $\partial M_L \cong \partial M_{L'}$ such that the following diagram commutes.*

$$\begin{tikzcd} H_1(M_L, P_L; \mathbb{Z}) \arrow[rr, "h_*", "\cong"'] \arrow[dr, "g_L"'] & & H_1(M_{L'}, P_{L'}; \mathbb{Z}) \arrow[dl, "g_{L'}"] \\ & H_1(M, P; \mathbb{Z}). & \end{tikzcd} \tag{1.0.3}$$

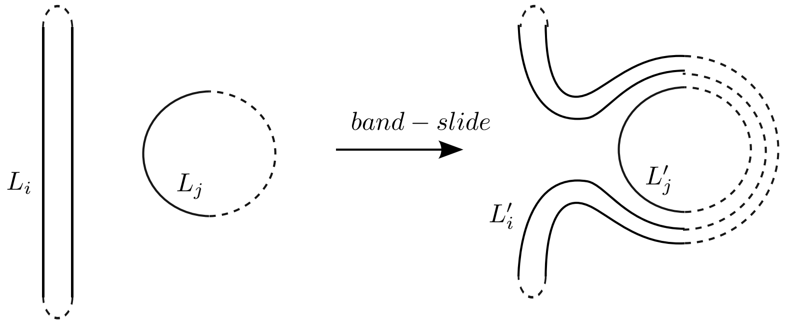



Figure 3: A band-slide of the component $L_i$ over $L_j$.

In a sequel we hope to be able to generalize Theorem 1.2 to $\mathbb{Z}$-null-homologous framed links in an arbitrary compact, connected, oriented 3–manifolds. For this result, we need new moves.

The second main result of this paper is a refinement of Theorem 1.2 to a special class of framed links, called *admissible framed links.*

Let $L = L_1 \cup \cdots \cup L_k$ be a $\mathbb{Z}$-null-homologous framed link in a compact, connected, oriented 3–manifold $M$. Note that $L$ admits a well-defined *linking matrix*

$$\mathrm{Lk}(L) = (l_{ij})_{1\le i,j\le k},$$

where $l_{ii}$ is the framing of $L_i$, and $l_{ij}$ $(i \neq j)$ is the linking number of $L_i$ and $L_j$. Then $L$ is said to be *admissible* if the linking matrix $\mathrm{Lk}(L)$ is diagonal with diagonal entries $\pm 1$. We call surgery along an admissible framed link an *admissible surgery.*

Admissible surgeries on 3–manifolds have been studied in several places. It is well known that every integral homology 3-sphere can be obtained from $S^3$ by an admissible surgery. Ohtsuki used admissible surgeries to define the notion of finite type invariants of integral homology spheres [16]. Cochran, Gerges and Orr [3] studied the equivalence relation on closed, oriented 3–manifolds generated by admissible surgeries, called 2*-surgeries* in [3]. They gave a characterization for two closed oriented 3–manifolds to be equivalent under admissible surgeries. Cochran and Melvin [4] used admissible surgeries to define finite type invariants of 3–manifolds generalizing Ohtsuki's definition of finite type invariants of integral homology spheres.

A *band-slide* on a framed link is an algebraically cancelling pair of handle-slides, see Figure 3. A band-slide preserves the homology classes of the components of a link , and also preserves the linking matrix of a $\mathbb{Z}$-null-homologous framed link. Thus, a band-slide on an admissible framed link yields an admissible framed link.

The first author proved the following refinement of Kirby's theorem to admissible framed links in $S^3$.

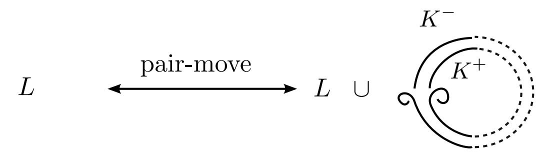



Figure 4: A pair-move

**Theorem 1.3** ([11]). *Let $L$ and $L'$ be two admissible framed links in $S^3$. Then the following conditions are equivalent.*

*(i) $L$ and $L'$ are related by a sequence of stabilizations and band-slides.*

*(ii) $S^3_L$ and $S^3_{L'}$ are orientation-preserving diffeomorphic.*

The second main result of this paper is Theorem 1.4 below, which refines Theorem 1.2 and generalizes Theorem 1.3. To state it, we need two new kinds of moves.

Two admissible framed links in $M$ are said to be related by a *pair move* if one of them, say $L$, is obtained from the other, say $L'$, by adjoining a 2-component admissible framed link $K^+ \cup K^-$ in $M \setminus L'$, where $K^+$ and $K^-$ are parallel to each other and $K^+$ and $K^-$ have framings $+1$ and $-1$, respectively, see 4. It follows that $L$ and $L'$ give diffeomorphic results of surgery, since one can handle-slide $K^+$ over $K^-$ to obtain from $L = L' \cup K^+ \cup K^-$ a framed link $\tilde{L} = L' \cup J \cup K^-$ which is related to $L'$ by a $\mathbb{Z}$-null-homologous $K_3$-move.

A *lantern-move* is defined as follows. Let $V_3$ be a handlebody of genus 3, which is identified with the complement of the tubular neighborhood of a trivial 3-component string link

$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 = (3 \text{ points}) \times [0,1]$$

in the cylinder $D^2 \times [0,1]$. Let $K$ and $K'$ be two framed links in $V_3$ as depicted in Figure 5(a) and (b), respectively. Here all components in $K$ and $K'$ are $+1$, where the framings are defined in the cylinder. Let $L$ be an admissible framed link in a 3–manifold $M$, and let $f\colon V_3 \hookrightarrow M \setminus L$ be an orientation-preserving embedding such that both $L \cup f(K)$ and $L \cup f(K')$ are admissible in $M$. (In fact, $L \cup f(K)$ is admissible if and only if $L \cup f(K')$ is admissible.) Then the two framed links $L \cup f(K)$ and $L \cup f(K')$ are said to be related by a lantern-move. A lantern-move preserves the diffeomorphism class of the results of surgery since we have a diffeomorphism $(V_3)_K \cong (V_3)_{K'}$ restricting to the canonical map $\partial(V_3)_K \cong \partial(V_3)_{K'}$. (The latter fact follows since the results from the framed string link $\gamma \subset D^2 \times [0,1]$ of surgeries along $K$ and $K'$ are equivalent. Alternatively, one can check that $K$ and $K'$ are $\delta$-equivalent in $V_3$.)

An *admissible IHX-move* is defined in Section 6.4

**Theorem 1.4.** *Let $M$, $P$ be as in Theorem 1.2. Let $L$ and $L'$ be admissible framed links in $M$. Then the following conditions are equivalent.*

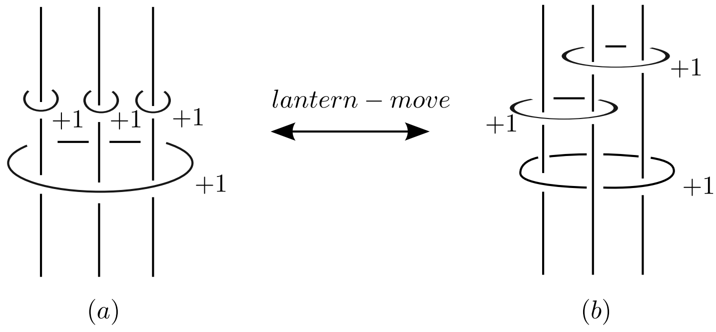



Figure 5: A lantern-move in a 3-handlebody.

*(i) $L$ and $L'$ are related by a sequence of stabilization, band-slides, pair-moves, admissible IHX-moves, and lantern-moves.*

*(ii) There is an orientation-preserving diffeomorphism $h\colon M_L \longrightarrow M_{L'}$ restricting to the natural identification map $\partial M_L \overset{\cong}{\to} \partial M_{L'}$ such that Diagram (1.0.3) commutes.*

## 2 Fenn-Rourke theorem for 3–manifolds with boundary

In this section, we state the generalization of Fenn and Rourke's theorem [6] to 3–manifolds with boundary that we proved in [12]. We mainly follow the constructions in [12], but the description is slightly simplified by the use of fundamental groupoids.

### 2.1 Fundamental groupoids

Let $X$ be a topological space, and let $P \subset X$ be a subset. Let $\Pi(X, P)$ denote the fundamental groupoid of $X$ with respect to $P$. The objects of $\Pi(X, P)$ are the elements of $P$, and the morphisms from $p \in P$ to $p' \in P$ are homotopy classes of paths from $p$ to $p'$. The set $\Pi(X, P)(p, p')$ of morphisms from $p$ to $p'$ is denoted usually by $\Pi(X; p, p')$. For $p \in P$, we set $\pi_1(X, p) = \Pi(X; p, p)$, the fundamental group of $X$ at $p$.

If $X$ is connected, then $\Pi(X, P)$ is a connected groupoid, i.e., for $p, p' \in P$, the set $\Pi(X; p, p')$ is non-empty. If, moreover, $N$ is a normal subgroup of $\pi_1(X, p)$, then we denote by $\Pi(X, P)/N$ the quotient of $\Pi(X, P)$ by the smallest congruence relation $\sim$ on $\Pi(X, P)$ such that $g \sim 1_p$ for all $g \in N$, where $1_p \in \pi_1(X; p)$ is the identity element.

In this paper, $\twoheadrightarrow$ for groupoids denotes an epimorphism in the category of groupoids, i.e., a full functor which is surjective on objects. However, all groupoid epimorphism denoted $\twoheadrightarrow$ will be bijective on objects. In the above situation we have an epimorphism

$$\Pi(X, P) \twoheadrightarrow \Pi(X, P)/N,$$

which is identity on objects.

In this section, we fix a compact, connected, oriented 3–manifold $M$ with *non-empty* boundary, whose components will be denoted $F_1, \ldots, F_t$ ($t \geq 1$). We also fix

$$P = \{p_1, \ldots, p_t\} \subset \partial M,$$

where $p_i \in F_i$ for each $i = 1, \ldots, t$. We consider the fundamental groupoid $\Pi(M, P)$ of $M$ with respect to $P$. Since $M$ is connected, the groups $\pi_1(M, p_i)$ for $i = 1, \ldots, t$ are isomorphic to each other. We regard $p_1$ as the basepoint of $M$, and often write $\pi_1 M = \pi_1(M, p_1)$.

## 2.2 Framed links and surgery

A *framed link* $L = L_1 \cup \cdots \cup L_n$ in $M$ is a link such that each component $L_i$ of $L$ is given a framing, i.e., a homotopy class of a simple closed curve $\gamma_i$ in the boundary $\partial N(L_i)$ of a tubular neighborhood $N(L_i)$ of $L_i$ in $M$ which is homotopic to $L_i$ in $N(L_i)$. *Surgery* along a framed link $L$ denotes the process of removing the interior of $N(L_i)$, and gluing a solid torus $D^2 \times S^1$ to $\partial N(L_i)$ so that the curve $\partial D^2 \times \{*\}$, $* \in S^1$, is attached to $\gamma_i \subset \partial N(L_i)$ for $i = 1, \ldots, n$. We denote the the result of surgery by $M_L$. Note that the boundary $\partial M_L$ is naturally identified with $\partial M$.

Surgery along a framed link can also be defined by using 4–manifolds. In the above situation, let $W_L$ denote the 4–manifold obtained from the cylinder $M \times I$, where $I = [0, 1]$, by attaching a 2-handle $D^2 \times D^2$ along $N(L_i) \times \{1\}$ using a diffeomorphism

$$S^1 \times D^2 \xrightarrow{\cong} N(L_i),$$

which maps $S^1 \times \{*\}$, $* \in \partial D^2$, onto the framing $\gamma_i$. We have a natural identification

$$\partial W_L \cong M \underset{\partial M}{\cup} (\partial M \times I) \underset{\partial M_L}{\cup} M_L,$$

Thus, $W_L$ is a cobordism between $M$ and $M_L$. Note that $\partial W_L$ is a connected, closed 3–manifold.

Set $P_L = \{p_1^L, \ldots, p_t^L\} \subset \partial M_L$, where $p_k^L = p_k \times \{1\} \in \partial M_L$ for $k = 1, \ldots, t$. Let $\gamma_k = p_k \times I \subset \partial W_L$ for $k = 1, \ldots, t$. Note that $\gamma_k$ is an arc in $\partial W$ from $p_k \in \partial M \subset \partial W_L$ to $p_k^L$.

The point $p_1$ is regarded as a basepoint of $W_L$ as well as of $M$, and we set $\pi_1 W_L := \pi_1(W_L, p_1)$. We regard $p_1^L$ as the basepoint of $M_L$ and write $\pi_1 M_L := \pi_1(M, p_1^L)$.

The inclusions

$$M \overset{i}{\hookrightarrow} W_L \overset{i'}{\hookleftarrow} M_L$$

induce full functors

$$\Pi(M,P) \xrightarrow{i_*} \Pi(W_L,P) \xleftarrow{i'_*} \Pi(M_L,P_L).$$

Here $i'_*$ is defined as the composite

$$\Pi(M_L,P_L) \xrightarrow{i'_*} \Pi(W_L,P_L) \xrightarrow[\cong]{\gamma_1,\dots,\gamma_t} \Pi(W_L,P),$$

where the second isomorphism is induced by the arcs $\gamma_1,\dots,\gamma_t$.

Let $N_L$ denote the normal subgroup of $\pi_1 M$ normally generated by the homotopy classes of the components of $L$. Then we have

$$N_L = \ker(i_*\colon\ \pi_1 M \to \pi_1 W_L). \tag{2.2.1}$$

## 2.3 Fenn-Rourke theorem for 3–manifolds with boundary

Fenn and Rourke [6, Theorem 6] characterized the condition for two framed links in a closed, oriented 3–manifold to be related by a sequence of stabilizations and handle-slides. The authors [12] generalized it to 3–manifolds with boundary. In this subsection we state this result in a slightly different way using fundamental groupoids.

Let $L$ and $L'$ be framed links in $M$ and suppose that there is an orientation-preserving diffeomorphism

$$h\colon\ M_L \to M_{L'}$$

which restricts to the natural identification map $\partial M_L \cong \partial M_{L'}$. Then we obtain a closed, oriented 4–manifold

$$W = W_{M,L,L',h} := W_L \cup_\partial (-W_{L'})$$

by gluing $W_L$ with $-W_{L'}$ along their boundaries using the map

$$h \cup \mathrm{id}_{(M\times\{0\})\cup(\partial M\times[0,1])}\colon\ \partial W_L \xrightarrow{\cong} \partial W_{L'}.$$

Suppose that we have $N_L = N_{L'}$. Then there exists a unique groupoid isomorphism $f\colon\ \Pi(W_L,P) \to \Pi(W_{L'},P)$, which is identity on objects, such that the triangle in the diagram

$$\begin{tikzcd}
\Pi(M_L,P_L) \arrow[rr, "h_*", "\cong"'] \arrow[d, "i'_*"'] & & \Pi(M_{L'},P_{L'}) \arrow[d, "i'_*"] \\
\Pi(W_L,P) \arrow[rr, "f", "\cong"'] & & \Pi(W_{L'},P) \\
& \Pi(M,P) \arrow[ul, "i_*"] \arrow[ur, "i_*"'] &
\end{tikzcd} \tag{2.3.1}$$

commutes.

Suppose that the square in Diagram (2.3.1) also commutes.

Let $j, j', u, u'$ be the inclusion maps in the diagram

$$\begin{tikzcd} \partial W_L \arrow[r, "u'"] \arrow[d, "u"'] & W_{L'} \arrow[d, "j'"] \\ W_L \arrow[r, "j"] & W. \end{tikzcd} \tag{2.3.2}$$

Consider the $\pi_1$ of the above diagram

$$\begin{tikzcd} \pi_1 \partial W_L \arrow[r, two heads, "u'_*"] \arrow[d, "u_*"'] & \pi_1 W_{L'} \arrow[d, "j'_*", "\cong"'] \\ \pi_1 W_L \arrow[r, "j_*", "\cong"'] \arrow[ur, "f_1", "\cong"'] & \pi_1 W. \end{tikzcd} \tag{2.3.3}$$

Here, the square is a pushout by the Van Kampen theorem since $\partial W_L$ is connected. The isomorphism $f_1$ is defined by

$$f_1 = f\colon\ \Pi(W_L, P)(p_1, p_1) \to \Pi(W_{L'}, P)(p_1, p_1).$$

In [12, Lemma 2.1], we proved that Diagram (2.3.3) commutes. It follows that $j_*$ and $j'_*$ are isomorphisms. Thus we have

$$\pi_1 W \cong \pi_1 W_L \cong \pi_1 W_{L'} \cong (\pi_1 M)/N_L.$$

Let $K(\pi_1 W, 1)$ be the Eilenberg–Mac Lane space, which is obtained from $W$ by adding cells of dimension $\geq 3$. Let

$$\rho_W\colon\ W \to K(\pi_1 W, 1)$$

be the inclusion map. We set

$$\eta(M, L, L', h) = (\rho_W)_*([W]) \in H_4(\pi_1 W),$$

where $[W] \in H_4 W$ is the fundamental class. Here, and in what follows, for a group $G$ we identify $H_*(K(G,1))$ with $H_*(G)$.

Now we state our generalization of Fenn and Rourke's theorem. (When $\partial M$ is connected, this is equivalent to the corresponding case of the statement given in [7, Theorem 4].)

**Theorem 2.1** ([12, Theorem 2.2]). *Let $L$ and $L'$ be framed links in a compact, connected, oriented 3–manifold $M$ with non-empty boundary. Then the following conditions are equivalent.*

*(i) $L$ and $L'$ are $\delta$-equivalent.*

*(ii) There is a diffeomorphism $h\colon M_L \xrightarrow{\cong} M_{L'}$ restricting the canonical identification map $\partial M_L \cong \partial M_{L'}$, and there is a groupoid isomorphism*

$$f\colon \Pi(W_L, P) \xrightarrow{\cong} \Pi(W_{L'}, P)$$

*such that Diagram (2.3.1) commutes and we have $\eta(M, L, L', h) = 0 \in H_4(\pi_1 W)$.*

Note that the statement of Theorem 2.1 is slightly different from [12, Theorem 2.2] in the following points:

- We use the fundamental groupoid $\Pi(M, P)$ etc. instead of $\pi_1(M, p_1, p_k)$ for $k = 1, \dots, t$.
- We use $\pi_1 W$ instead of $\pi_1 W_L (\cong \pi_1 W)$.

These differences are not essential, and one can easily check that Theorem 2.1 is equivalent to [12, Theorem 2.2].

# 3 $N$-links

In this section, we give a modification of Theorem 2.1 which will be useful in many situations.

We fix a normal subgroup $N$ of $\pi_1 M$. Let

$$q\colon \pi_1 M \twoheadrightarrow \pi_1 M/N$$

denote the projection, which naturally extends to a full functor

$$q\colon \Pi(M, P) \twoheadrightarrow \Pi(M, P)/N.$$

## 3.1 $N$-links and surgery

A framed link $L$ in $M$ is called an *$N$-link* in $M$ if $N_L \subset N$, i.e., if the homotopy class of each component of $L$ is in $N$.

For an $N$-link $L$ in $M$, consider the following diagram

$$\begin{tikzcd}
\pi_1 M_L \arrow[r, two heads, "i'_*"] \arrow[drr, two heads, "q_L"'] & \pi_1 W_L \arrow[dr, two heads, "\bar{q}_L"] & \pi_1 M \arrow[l, two heads, "i_*"'] \arrow[d, two heads, "q"] \\
 & & \pi_1 M/N.
\end{tikzcd} \tag{3.1.1}$$

Since $N_L \subset N$, there is a unique surjective homomorphism $\bar{q}_L$ such that $q = \bar{q}_L i_*$. We set $q_L := \bar{q}_L i'_*$. Diagram (3.1.1) naturally extends to a commutative diagram in groupoids

$$\begin{tikzcd}
\Pi(M_L, P_L) \arrow[r, two heads, "i'_*"] \arrow[drr, two heads, "q_L"'] & \Pi(W_L, P) \arrow[dr, two heads, "\bar{q}_L"] & \Pi(M, P) \arrow[l, two heads, "i_*"'] \arrow[d, two heads, "q"] \\
 & & \Pi(M, P)/N.
\end{tikzcd} \tag{3.1.2}$$

Suppose that $L$ and $L'$ are $N$-links in $M$ and $h\colon\, M_L \overset{\cong}{\to} M_{L'}$ is a diffeomorphism restricting to the canonical map $\partial M_L \cong \partial M_{L'}$ such that the following diagram commutes.

$$
\begin{tikzcd}
\Pi(M_L, P_L) \arrow[rr, "h_*", "\cong"'] \arrow[dr, two heads, "q_L"'] & & \Pi(M_{L'}, P_{L'}) \arrow[dl, two heads, "q_{L'}"] \\
& \Pi(M,P)/N. &
\end{tikzcd}
\tag{3.1.3}
$$

**Lemma 3.1.** *In the above situation, there is a unique functor*

$$g\colon\, \Pi(W,P) \twoheadrightarrow \Pi(M,P)/N$$

*such that* $\bar{q}_L = gj_*$ *and* $\bar{q}_{L'} = gj'_*$.

*Proof.* Consider the following diagram.

$$
\begin{tikzcd}
\Pi(M_L, P_L) \arrow[rrrr, bend left, "h_*"] \arrow[dd, "(i')^L_*"'] \arrow[drr, "v_*"] \arrow[ddddrr, bend right=60, "q_L"'] & & \Pi(M,P) \arrow[d, "k_*"] \arrow[ddll, "i^L_*"'] \arrow[ddrr, "i^{L'}_*"] & & \Pi(M_L, P_L) \arrow[dll, "v'_*"'] \arrow[dd, "(i')^{L'}_*"] \arrow[ddddll, bend left=60, "q_{L'}"] \\
& & \Pi(\partial W_L, P) \arrow[dll, "u_*"] \arrow[drr, "u'_*"'] & & \\
\Pi(W_L, P) \arrow[drr, "j_*"] \arrow[ddrr, "\bar{q}_L"'] & & & & \Pi(W_{L'}, P) \arrow[dll, "j'_*"'] \arrow[ddll, "\bar{q}_{L'}"] \\
& & \Pi(W,P) \arrow[d, dotted, "g"] & & \\
& & \Pi(M,P)/N & &
\end{tikzcd}
\tag{3.1.4}
$$

The arrows above $\Pi(W,P)$ are induced by a commutative diagram of inclusions of submanifolds of $W$, and hence commute. The middle diamond $j_*u_* = j'_*u'_*$ is a pushout. Therefore, to prove existence of $g\colon\, \Pi(W,P) \to \Pi(M,P)/N$ which makes the above diagram commute (i.e., $gj_* = \bar{q}_L$ and $gj'_* = \bar{q}_{L'}$), it suffices to prove that $\bar{q}_L u_* = \bar{q}_{L'} u'_*$. Since the groupoid $\Pi(\partial W_L, P)$ is generated by the images of $k_*$ and $v_*$, it suffices to check that

$$\bar{q}_L u_* k_* = \bar{q}_{L'} u'_* k_*, \quad \bar{q}_L u_* v_* = \bar{q}_{L'} u'_* v_*$$

Indeed, we have

$$\bar{q}_L u_* k_* = \bar{q}_L i^L_* = \bar{q}_{L'} i^{L'}_* = \bar{q}_{L'} u'_* k_*,$$

and

$$\bar{q}_L u_* v_* = \bar{q}_L (i')^L_* = q_L = q_{L'} h_* = \bar{q}_{L'} (i')^{L'}_* h_* = \bar{q}_{L'} u'_* v'_* h_* = \bar{q}_{L'} u'_* v_*.$$

□

By Lemma 3.1, there is a surjective homomorphism

$$g\colon \pi_1 W \twoheadrightarrow \pi_1 M/N.$$

Let $K(\pi_1 M/N, 1)$ be obtained from $K(\pi_1 W, 1)$ by attaching cells of dimension $\geq 2$. Let

$$\rho_{W,N}\colon W \to K(\pi_1 M/N, 1)$$

be the inclusion map. Now, define a homology class $\eta_{\pi_1 M/N}(M, L, L', h) \in H_4(\pi_1 M/N)$ by

$$\eta_{\pi_1 M/N}(M, L, L', h) = (\rho_{W,N})_*([W]).$$

## 3.2 $K_3(N)$-moves

A $K_3(N)$*-move* on a framed link in $M$ is a $K_3$-move $L \leftrightarrow L \cup K \cup K'$ as in Figure 2, where the homotopy class of $K$ is contained in $N$. Note that a $K_3(N)$-move on an $N$-link produces another $N$-link.

The $\delta(N)$*-equivalence* on $N$-links in $M$ is defined as the equivalence relation generated by stabilization, handle-slides and $K_3(N)$-moves.

Suppose that $L$ and $L \cup K \cup K'$ are related by a $K_3(N)$-move as above. Let $V$ be a tubular neighborhood of $K$ in $M \setminus L$ containing $K'$ in the interior. Then there is a diffeomorphism $h\colon M_L \cong M_{L \cup K \cup K'}$ restricting to the identity on $M \setminus \operatorname{int} V$. Such an $h$ is unique up to isotopy fixing $M \setminus \operatorname{int} V$. The 4–manifold $W_{L \cup K \cup K'}$ is diffeomorphic to the 4–manifold obtained from $W_L$ by surgery along the framed knot

$$\tilde{K} := K \times \{1/2\} \subset M \times [0,1] \subset W_L \,(= M \times [0,1] \cup \text{(2-handles)}).$$

Here, the framing of $\tilde{K}$ is determined by that of $K$. Thus, there is a natural surjective homomorphism

$$\theta\colon \pi_1 W_L \twoheadrightarrow \pi_1 W_{L \cup K \cup K'}$$

with kernel normally generated by the homotopy class of $\tilde{K}$. We have a cobordism

$$X := (W_L \times [0,1]) \cup (\text{2-handles attached along } \tilde{K} \times \{1\})$$

between $W_L$ and $(W_L)_{\tilde{K}} \cong W_{L \cup K \cup K'}$. This cobordism $X$ is over $K(\pi_1 M/N, 1)$ since $\tilde{K}$ maps to null-homotopic loop in $K(\pi_1 M/N, 1)$.

The homomorphism $\theta$ extends in a natural way to a full, identity-on-objects functor

$$\theta\colon \Pi(W_L, P) \twoheadrightarrow \Pi(W_{L \cup K \cup K'}, P).$$

**Lemma 3.2.** *In the above situation with $L' := L \cup K \cup K'$, Diagram (3.1.3) commutes and we have*

$$\eta_{\pi_1 M/N}(M, L, L', h) = 0 \in H_4(\pi_1 M/N). \tag{3.2.1}$$

*Proof.* To prove commutativity of (3.1.3), consider the following diagram.

$$\begin{tikzcd}
\Pi(M_L, P_L) \arrow[rr, two heads, "h_*", "\cong"'] \arrow[d, two heads, "(i')^L_*"] \arrow[ddd, bend right=60, two heads, "q_L"'] & & \Pi(M_{L'}, P_{L'}) \arrow[d, two heads, "(i')^{L'}_*"'] \arrow[dddll, bend left=60, two heads, "q_{L'}"] \\
\Pi(W_L, P) \arrow[rr, two heads, "\theta"] \arrow[dddr, two heads, "\bar{q}_L"'] & & \Pi(W_{L'}, P) \arrow[dddl, two heads, "\bar{q}_{L'}"] \\
& \Pi(M, P) \arrow[ul, two heads, "i^L_*"'] \arrow[ur, two heads, "i^{L'}_*"] \arrow[d, two heads, "q"] & \\
& \Pi(M, P)/N &
\end{tikzcd} \tag{3.2.2}$$

We have

$$q_{L'} h_* = \bar{q}_{L'} (i')^{L'}_* h_* = \bar{q}_{L'} \theta (i')^L_* = \bar{q}_L (i')^L_* = q_L.$$

Here we have $\bar{q}_{L'} \theta = \bar{q}_L$ since we have

$$\bar{q}_{L'} \theta i^L_* = \bar{q}_{L'} i^{L'}_* = q = \bar{q}_L i^L_*$$

and the functor $i^L_*$ is full and identity on objects.

Now, we will prove (3.2.1). As we have observed, $W_L$ and $W_{L'} = W_{L \cup K \cup K'}$ are cobordant over $K(\pi_1 M/N, 1)$. Hence we have

$$\eta_{\pi_1 M/N}(M, L, L', h) = (\rho_{W,N})_*([W]) = 0.$$

□

## 3.3 Characterization of $\delta(N)$-equivalence

We have the following characterization of the $\delta(N)$-equivalence.

**Theorem 3.3.** *Let $M$ be a compact, connected, oriented 3–manifold with non-empty boundary. Let $P \subset \partial M$ contain exactly one point of each connected component of $\partial M$. Let $N$ be a normal subgroup of $\pi_1 M$. Let $L$ and $L'$ be $N$-links in $M$. Then the following conditions are equivalent.*

*(i) $L$ and $L'$ are $\delta(N)$-equivalent.*

*(ii) There is a diffeomorphism $h\colon M_L \cong M_{L'}$ restricting to the canonical map $\partial M_L \cong \partial M_{L'}$ such that Diagram (3.1.3) commutes and we have*

$$\eta_{\pi_1 M/N}(M, L, L', h) = 0 \in H_4(\pi_1 M/N). \tag{3.3.1}$$

*Proof of the "only if" part of Theorem 3.3.* This part follows from Lemma 3.2 and the "only if" part of Theorem 2.1. $\square$

For the "if part", we first consider the case where $N$ is normally finitely generated in $\pi_1 M$.

*Proof of the "if" part of Theorem 3.3. where $N$ is normally finitely generated in $\pi_1 M$.* By the assumption, there is a framed link $K = K_1 \cup \cdots \cup K_k$ in $M$ disjoint from both $L$ and $L'$ such that $N_K = N$. Let $K^* = K_1^* \cup \cdots \cup K_k^*$ be a framed link in $M$ consisting of small 0-framed meridians $K_j^*$ to $K_j$. Thus $L$ and $\tilde{L} := L \cup K \cup K^*$ (resp. $L'$ and $\tilde{L}' := L' \cup K \cup K^*$) are related by $k$ $K_3(N)$-moves. We have $N = N_{\tilde{L}} = N_{\tilde{L}'}$.

It suffices to prove that $\tilde{L}$ and $\tilde{L}'$ are $\delta$-equivalent. Consider the following diagram.

$$\begin{tikzcd}
\Pi(M_L, P_L) \arrow[r, "m_*", "\cong"'] \arrow[rrrr, bend left, "h_*", "\cong"'] \arrow[d, "(i')^L_*"] \arrow[dddrr, bend right, two heads, "q_L"'] & \Pi(M_{\tilde{L}}, P_{\tilde{L}}) \arrow[rr, "\tilde{h}_* = (m' h m^{-1})_*", "\cong"'] \arrow[d, "(i')^{\tilde{L}}_*"'] & & \Pi(M_{\tilde{L}'}, P_{\tilde{L}'}) \arrow[d, "(i')^{\tilde{L}'}_*"'] & \Pi(M_{L'}, P_{L'}) \arrow[l, "m'_*"', "\cong"] \arrow[d, "(i')^{L'}_*"'] \arrow[dddll, bend left, two heads, "q_{L'}"] \\
\Pi(W_L, P) \arrow[r, "\theta", "\cong"'] \arrow[ddrr, "\bar{q}_L"'] & \Pi(W_{\tilde{L}}, P) \arrow[rr, "f := \bar{q}_{\tilde{L}'}^{-1} \bar{q}_{\tilde{L}}", "\cong"'] \arrow[ddr, "\bar{q}_{\tilde{L}}"'] & & \Pi(W_{\tilde{L}'}, P) \arrow[ddl, "\bar{q}_{\tilde{L}'}"] & \Pi(W_{L'}, P) \arrow[l, "\theta'"', "\cong"] \arrow[ddll, "\bar{q}_{L'}"] \\
& & \Pi(M, P) \arrow[ul, "i^{\tilde{L}}_*"', "\cong"] \arrow[ur, "i^{\tilde{L}'}_*", "\cong"'] \arrow[d, "q"] & & \\
& & \Pi(M, P)/N & &
\end{tikzcd} \tag{3.3.2}$$

Here $m\colon M_L \xrightarrow{\cong} M_{\tilde{L}}$ and $m'\colon M_{L'} \xrightarrow{\cong} M_{\tilde{L}'}$ are natural diffeomorphisms, and we set $\tilde{h} = m' h m^{-1}\colon M_{\tilde{L}} \xrightarrow{\cong} M_{\tilde{L}'}$. All the faces except the middle square commute. Since the outermost triangle commutes, i.e., $q_L = q_{L'} h_*$, one can check that

$$\bar{q}_{\tilde{L}'} f (i')^{\tilde{L}}_* = \bar{q}_{\tilde{L}'} (i')^{\tilde{L}'}_* \tilde{h}_*.$$

Since $\bar{q}_{\tilde{L}'}$ is an isomorphism, the middle square commutes, i.e.,

$$f (i')^{\tilde{L}}_* = (i')^{\tilde{L}'}_* \tilde{h}_*.$$

Thus, the whole Diagram (3.3.2) commutes.

Set

$$W := W_{M,L,L',h} = W_L \cup_\partial (-W_{L'}),$$
$$\tilde{W} := W_{M,\tilde{L},\tilde{L}',\tilde{h}} = W_{\tilde{L}} \cup_\partial (-W_{\tilde{L}'}).$$

By commutativity of the middle pentagon, the homology class

$$\eta_{\pi_1 M/N}(M, \tilde{L}, \tilde{L}', \tilde{h}) = (\rho_{\tilde{W}})_*([\tilde{W}]) \in H_4(\pi_1 M/N),$$

is defined. We claim that $\tilde{W}$ and $W$ are bordant over $K(\pi_1 M/N)$. Indeed, there is an oriented, compact 5-cobordism $X$ between $W$ and $\tilde{W}$ constructed as in Section 3.2, which maps to $K(\pi_1 M/N, 1)$. This implies (3.3.1) since we have

$$\eta_{\pi_1 M/N}(M, L, L', h) = 0 \in H_4(\pi_1 M/N).$$

Then, by Theorem 2.1, it follows that $\tilde{L}$ and $\tilde{L}'$ are $\delta$-equivalent. $\square$

*Proof of the "if" part of Theorem 3.3, general case.* Let $N_0 \subset N$ denote the smallest normal subgroup in $\pi_1 M$ containing $N_L \cup N_{L'}$. Let

$$q^0\colon\ \Pi(M, P) \twoheadrightarrow \Pi(M, P)/N_0$$

be the projection. Let

$$\bar{q}^0_L\colon\ \Pi(W_L, P) \twoheadrightarrow \Pi(M, P)/N_0$$

be the homomorphism such that $q^0 = \bar{q}^0_L i^L_*$. Set

$$q^0_L = \bar{q}^0_L (i')^L_*\colon\ \Pi(M_L, P_L) \twoheadrightarrow \Pi(M, P)/N_0.$$

Similarly, define

$$\bar{q}^0_{L'}\colon\ \Pi(W_{L'}, P) \twoheadrightarrow \Pi(M, P)/N_0$$

and

$$q^0_L\colon\ \Pi(M_{L'}, P_{L'}) \twoheadrightarrow \Pi(M, P)/N_0.$$

Let $\bar{N}_1 \subset \pi_1 M/N_0$ be the normal subgroup generated by the elements

$$q^0_L(a)^{-1} q^0_{L'}(h_*(a))$$

for $a \in \pi_1 M_L$. By $q_L = q_{L'} h_*$, it follows that $\bar{N}_1 \subset N/N_0$. Since $\pi_1 M_L$ is finitely generated, it follows that $\bar{N}_1$ is finitely normally generated in $\pi_1 M/N_0$. Set

$$N_1 = (q^0)^{-1}(\bar{N}_1) \subset N,$$

which is finitely normally generated in $\pi_1 M$.

Let $p_{N_0,N_1}\colon\ \Pi(M, P)/N_0 \twoheadrightarrow \Pi(M, P)/N_1$ be the projection. Set

$$q^1_L = p_{N_0,N_1} q^0_L\colon\ \Pi(M_L, P_L) \twoheadrightarrow \Pi(M, P)/N_1,$$
$$q^1_{L'} = p_{N_0,N_1} q^0_{L'}\colon\ \Pi(M_{L'}, P_{L'}) \twoheadrightarrow \Pi(M, P)/N_1.$$

We have $q^1_L = q^1_{L'} h_*$. Hence we have a well-defined homology class

$$\eta_{\pi_1 M/N_1}(M, L, L', h) \in H_4(\pi_1 M/N_1).$$

Since $N$ is a union of finitely normally generated subgroups of $\pi_1 M$ and homology preserves direct limits, it follows that there is a normally finitely generated subgroup $N_2$ of $\pi_1 M$ such that $N_1 \subset N_2 \subset N$ and

$$(p_{N_1,N_2})_*(\eta_{\pi_1 M/N_1}(M, L, L', h)) = \eta_{\pi_1 M/N_2}(M, L, L', h) = 0 \in H_4(\pi_1 M/N_2),$$

where $p_{N_1,N_2}\colon \Pi(M,P)/N_1 \twoheadrightarrow \Pi(M,P)/N_2$ is the projection. The following triangle commutes

$$\begin{tikzcd}
\Pi(M_L, P_L) \arrow[rr, "h_*", "\cong"'] \arrow[dr, two heads, "q_L^2"'] & & \Pi(M_{L'}, P_{L'}) \arrow[dl, two heads, "q_{L'}^2"] \\
& \Pi(M,P)/N_2, &
\end{tikzcd}$$

where $q_L^2 = p_{N_1,N_2} q_L^1$ and $q_{L'}^2 = p_{N_1,N_2} q_{L'}^1$. Now we can apply the above-proved case of the theorem to deduce that $L$ and $L'$ are $\delta(N_2)$-equivalent. Hence they are $\delta(N)$-equivalent. $\square$

# 4 Manifolds over $K(G, 1)$

## 4.1 Bordism groups

Fix a group $G$. Let $K(G, 1)$ denote the Eilenberg–Mac Lane space.

By an *$n$–manifold over $K(G, 1)$* or *$G$-$n$–manifold* we mean a pair $(M, \rho_M)$ of a compact, oriented, smooth $n$–manifold $M$ and a map $\rho_M\colon M \to K(G, 1)$. Here we require no condition about the basepoints even when $M$ has a specified basepoint. A $G$-$n$–manifold $(M, \rho_M)$ will often be simply denoted by $M$.

For $n \geq 0$, let $\Omega_n(G) = \Omega_n(K(G, 1))$ denote the $n$-dimensional oriented bordism group of $K(G, 1)$, which is defined to be the set of bordism classes of closed $G$-$n$–manifolds.

There is a natural map

$$\theta_n\colon \Omega_n(G) \to H_n(G)$$

defined by

$$\theta_n([M, \rho_M]) = (\rho_M)_*([M]) \in H_n(G).$$

It is known that $\theta_n$ is an isomorphism for $n = 1, 2, 3$. For $n = 4$ we have an isomorphism

$$\begin{pmatrix} \theta_4 \\ \sigma \end{pmatrix}\colon \Omega_4(G) \xrightarrow{\cong} H_4(G) \oplus \mathbb{Z}. \tag{4.1.1}$$

where

$$\sigma([M, \rho_M]) = \text{signature}(M) \in \mathbb{Z}.$$

## 4.2 $G$-surfaces, bordered $G$-3–manifolds and cobordisms

A $G$-2–manifold is called a $G$*-surface*.

Let $(\Sigma, \rho_\Sigma)$ be a $G$-surface. A $(\Sigma, \rho_\Sigma)$*-bordered* 3*–manifold* will mean a triple $(M, \rho_M, \phi_M)$ such that $(M, \rho_M)$ is a $G$-3–manifold and $\phi_M : \Sigma \overset{\cong}{\to} \partial M$ is an orientation-preserving diffeomorphism satisfying $\rho_\Sigma = (\rho_M|_{\partial M})\phi_M$.

A *cobordism* between two $(\Sigma, \rho_\Sigma)$-bordered 3–manifolds $(M, \rho_M, \phi_M)$ and $(M', \rho_{M'}, \phi_{M'})$ is a triple $(W, \rho_W, \phi_W)$ consisting of a $G$-4–manifold $(W, \rho_W)$ and an orientation-preserving diffeomorphism

$$\phi_W : M \cup_\Sigma (-M') \overset{\cong}{\to} \partial W,$$

where $M \cup_\Sigma (-M')$ is the closed oriented 4–manifold obtained by gluing $M$ and $-M'$ along their boundaries using the diffeomorphism $\phi_{M'}\phi_M^{-1} : \partial M \overset{\cong}{\to} \partial M'$, such that the following diagram commutes

$$\begin{tikzcd}
M \arrow[d, "\mathrm{incl}"'] \arrow[drr, "\rho_M"] & & \\
M \cup_\Sigma (-M') \arrow[r, "\phi_W"] & W \arrow[r, "\rho_W"] & K(G,1). \\
M' \arrow[u, "\mathrm{incl}"] \arrow[urr, "\rho_{M'}"'] & &
\end{tikzcd}$$

We denote this situation by $(W, \rho_W, \phi_W) : (M, \rho_M, \phi_M) \to (M', \rho_{M'}, \phi_{M'})$ or simply by $W : M \to M'$.

Two cobordisms $W, W' : M \to M'$ between $(\Sigma, \rho_\Sigma)$-bordered 3–manifolds $M = (M, \rho_M)$ and $M' = (M', \rho_{M'})$ are said to be *cobordant* if there is a *cobordism* between them, i.e. a triple $(X, \rho_X, \phi_X)$ consisting of a $G$-5–manifold $X = (X, \rho_X)$ and an orientation-preserving diffeomorphism

$$\phi_X : W'' \overset{\cong}{\to} \partial X,$$

where $W'' := W \cup_{M \cup_\Sigma (-M')} (-W')$ is the closed, oriented 4–manifold obtained from $W$ and $-W'$ by gluing along $M \cup_\Sigma (-M')$ using the diffeomorphism $\phi_{W'}\phi_W^{-1} : W \to W'$, such that the following diagram commutes

$$\begin{tikzcd}
W \arrow[d, "\mathrm{incl}"'] \arrow[drr, "\rho_W"] & & \\
W'' \arrow[r, "\phi_X"] & X \arrow[r, "\rho_X"] & K(G,1). \\
W' \arrow[u, "\mathrm{incl}"] \arrow[urr, "\rho_{W'}"'] & &
\end{tikzcd}$$

## 4.3 Cobordism groupoid $\mathcal{C} = \mathcal{C}_{(\Sigma,\rho_\Sigma)}$

As in the last subsection, let $\Sigma = (\Sigma, \rho_\Sigma)$ be a $G$-surface.

For our purpose, it is convenient to introduce the category $\mathcal{C} = \mathcal{C}_{(\Sigma,\rho_\Sigma)}$ of $\Sigma$-bordered 3–manifolds and cobordism classes of cobordisms between $\Sigma$-bordered 3–manifolds, defined as follows.

The objects in $\mathcal{C}$ are $\Sigma$-bordered 3–manifolds. The morphisms between two $\Sigma$-bordered 3–manifolds $M = (M, \rho_M, \phi_M)$ and $M' = (M', \rho_{M'}, \phi_{M'})$ are the cobordism classes of the cobordisms between $M$ and $M'$.

The composition in $\mathcal{C}$ is induced by the composition of cobordisms defined below. Two cobordisms $W\colon M \to M'$ and $W'\colon M' \to M''$ can be composed in the usual way. Let $W' \circ W = W' \cup_{M'} W$ be the 4–manifold obtained by gluing $W'$ and $W$ along $M'$ using the map

$$\phi_W(M') \underset{\cong}{\xrightarrow{(\phi_W|_{M'})^{-1}}} M' \underset{\cong}{\xrightarrow{\phi_{W'}|_{M'}}} \phi_{W'}(M').$$

Let

$$\rho_{W'\circ W} = \rho_{W'} \cup \rho_W\colon W' \circ W \to K(G,1),$$

and

$$\phi_{W'\circ W} = (\phi_{W'}|_{M''}) \cup (\phi_W|_M)\colon M \cup_\Sigma (-M'') \overset{\cong}{\to} \partial(W' \circ W).$$

Then we obtain a new cobordism

$$W' \circ W = (W' \circ W, \rho_{W'\circ W}, \phi_{W'\circ W})\colon (M, \rho_M, \phi_M) \to (M'', \rho_{M''}, \phi_{M''}).$$

The identity morphism $1_M\colon M \to M$ is represented by the "reduced" cylinder $C_M = (C_M, \rho_{C_M}, \phi_{C_M})$. The 4–manifold $C_M$ is defined by

$$C_M = M \times [0,1]/\sim, \tag{4.3.1}$$

where $\sim$ is generated by $(x,t) \sim (x,t')$ for $x \in \partial M$ and $t, t' \in [0,1]$. The map $\rho_{C_M}\colon C_M \to K(G,1)$ is induced by the composite

$$M \times [0,1] \xrightarrow{\text{proj}} M \xrightarrow{\rho_M} K(G,1).$$

The map $\phi_{C_M}\colon M \cup_\Sigma (-M) \to \partial C_M$ is given by

$$\phi_{C_M} = \phi_{M,\partial C_M} \cup \phi_{-M,\partial C_M},$$

where $\phi_{M,\partial C_M}\colon M \hookrightarrow \partial C_M$ is induced by $M \cong M \times \{1\} \hookrightarrow M \times [0,1]$, and $\phi_{-M,\partial C_M}\colon (-M) \hookrightarrow \partial C_M$ is induced by $M \cong M \times \{0\} \hookrightarrow M \times [0,1]$.

It is not difficult to check that the above definition gives a well-defined category.

By abuse of notation, the morphism in $\mathcal{C}$ represented by a cobordism $W = (W, \rho_W, \phi_W)$ from $M$ to $M'$ is again denoted by $W = (W, \rho_W, \phi_W)$.

The category $\mathcal{C}$ is a groupoid by the same reason that $\Omega_n(G)$ is a group. Indeed, for a morphism $W = (W, \rho_W, \phi_W)\colon (M, \rho_M, \phi_M) \to (M', \rho_{M'}, \phi_{M'})$ in $\mathcal{C}$, the inverse $W^{-1}$ is represented by the cobordism

$$W^{-1} := (-W, \rho_{W^{-1}}, \phi_{W^{-1}})\colon (M', \rho_{M'}, \phi_{M'}) \to (M, \rho_M, \phi_M)$$

where $\rho_{W^{-1}} = \rho_W\colon (-W) \to K(G,1)$, and $\phi_{W^{-1}}\colon M' \cup_\Sigma (-M) \overset{\cong}{\to} \partial(-W)$ is the composite

$$M' \cup_\Sigma (-M) \cong -(M \cup_\Sigma (-M)) \underset{\cong}{\xrightarrow{-\phi_W}} (-\partial W) \cong \partial(-W).$$

The composite $W^{-1} \circ W$ is cobordant to $C_M$ via the cobordism $(X, \rho_X, \phi_X)$, where the 5–manifold $X$ is the "partially reduced cylinder"

$$X := (W \times [0,1])/((\phi_W(x), t) \sim (\phi_W(x), t') \text{ for } x \in (-M') \subset M \cup_\partial (-M'),\ t \in [0,1]),$$

the map $\rho_X\colon X \to K(G,1)$ is induced by the composite

$$W \times [0,1] \xrightarrow{\text{proj}} W \xrightarrow{\rho_W} K(G,1),$$

and the diffeomorphism

$$\phi_X\colon (W^{-1} \circ W) \cup_\partial (-C_M) \to \partial X$$

is given by

$$\begin{aligned} \phi_X(w) &= (w, 0) \quad \text{for } w \in W \subset W^{-1} \circ W, \\ \phi_X(w) &= (w, 1) \quad \text{for } w \in -W \subset W^{-1} \circ W, \\ \phi_X([x,t]) &= (i_{M,W}, t) \quad \text{for } x \in M,\ t \in [0,1]. \end{aligned}$$

Here $[x,t] \in C_M$ is represented by $(x,t) \in M \times [0,1]$, and $i_{M,W}\colon M \to W$ is the composite

$$M \subset M \cup_\partial (-M') \xrightarrow{\phi_W} \partial W \subset W.$$

Similarly, $W \circ W^{-1}$ is cobordant to $C_{M'}$. Thus $W\colon M \to M'$ is an isomorphism in $\mathcal{C}$.

## 4.4 $G$-diffeomorphism

Let $(M, \rho_M, \phi_M)$ and $(M', \rho_{M'}, \phi_{M'})$ be two $(\Sigma, \rho_\Sigma)$-bordered 3–manifolds.

By a *$G$-diffeomorphism*

$$h\colon (M, \rho_M, \phi_M) \overset{\cong}{\to} (M', \rho_{M'}, \phi_{M'})$$

we mean a diffeomorphism $h\colon M \overset{\cong}{\to} M'$ such that

(i) $h$ is compatible with the maps $\phi_M\colon \Sigma \overset{\cong}{\to} \partial M$ and $\phi_{M'}\colon \Sigma \overset{\cong}{\to} \partial M'$, i.e., we have $\phi_{M'} = (h|_{\partial M})\phi_M$,

(ii) $h$ is compatible with the maps $\rho_M\colon M \to K(G,1)$ and $\rho_{M'}\colon M' \to K(G,1)$ *up to homotopy*, i.e., we have

$$\rho_M \simeq \rho_{M'}h\colon M \to K(G,1), \tag{4.4.1}$$

where $\simeq$ denotes homotopy through maps $M \to K(G,1)$ restricting to $\rho_M|_{\partial M}$.

In this case, $(M,\rho_M,\phi_M)$ and $(M',\rho_{M'},\phi_{M'})$ are said to be *$G$-diffeomorphic.*

We have the following characterization of $G$-diffeomorphism in terms of fundamental groupoids.

**Proposition 4.1.** *Let $(\Sigma,\rho_\Sigma)$ be a non-empty $G$-surface. Let $(M,\rho_M,\phi_M)$ and $(M',\rho_{M'},\phi_{M'})$ be connected $(\Sigma,\rho_\Sigma)$-bordered 3–manifolds. Let $P_\Sigma \subset \Sigma$ be a subset containing exactly one point of each connected component of $\Sigma$, and set $P = \phi_M(P_\Sigma) \subset \partial M$ and $P' = \phi_{M'}(P_\Sigma) \subset \partial M'$. Then the following conditions are equivalent.*

*(i) $(M,\rho_M,\phi_M)$ and $(M',\rho_{M'},\phi_{M'})$ are $G$-diffeomorphic.*

*(ii) There is a diffeomorphism $h\colon M \overset{\cong}{\to} M'$ compatible with the maps $\phi_M$ and $\phi_{M'}$ such that the following groupoid diagram commutes*

$$\begin{tikzcd}
\Pi(M,P) \arrow[rr, "h_*", "\cong"'] \arrow[dr, "(\rho_M)_*"'] & & \Pi(M',P') \arrow[dl, "(\rho_{M'})_*"] \\
& \Pi(K(G,1),\rho_\Sigma(P_\Sigma)). &
\end{tikzcd} \tag{4.4.2}$$

*Proof.* It is clear that (i) implies (ii).

Suppose that (ii) holds. It suffices to prove (4.4.1). Suppose $p_1,\dots,p_t$ ($t \geq 1$) be the elements of $P_\Sigma$. For $i = 2,\dots,t$, let $\gamma_i$ be a simple curve between $\phi_M(p_1)$ and $\phi_M(p_i)$ in $M$ such that $\gamma_i \cap \gamma_j = \{\phi_M(p_1)\}$ if $i \neq j$. Commutativity of (4.4.2) implies that, for each $i = 2,\dots,t$, the maps $\rho_M|_{\gamma_i}\colon \gamma_i \to K(G,1)$ and $(\rho_{M'}h)|_{\gamma_i}\colon \gamma_i \to K(G,1)$ are homotopic rel endpoints. Hence $\rho_M$ is homotopic fixing $\partial M$ to a map $(\rho_M)_1\colon M \to K(G,1)$ such that

$$(\rho_M)_1|_{\partial M\cup\gamma_2\cup\cdots\cup\gamma_t} = (\rho_{M'}h)|_{\partial M\cup\gamma_2\cup\cdots\cup\gamma_t}.$$

Note that the subcomplex $\partial M \cup \gamma_2 \cup \cdots \cup \gamma_t$ of $M$ is connected. By (4.4.2) we have the following commutative diagram.

$$\begin{tikzcd}
\pi_1(M,\phi_M(p_1)) \arrow[rr, "h_*", "\cong"'] \arrow[dr, "(\rho_M)_*"'] & & \pi_1(M',\phi_{M'}(p_1)) \arrow[dl, "(\rho_{M'})_*"] \\
& \pi_1(K(G,1),\rho_\Sigma(p_1)) = G. &
\end{tikzcd} \tag{4.4.3}$$

By the property of the Eilenberg–Mac Lane space $K(G,1)$, it follows that $(\rho_M)_1$ is homotopic fixing $\partial M \cup \gamma_2 \cup \cdots \cup \gamma_t$ to $\rho_{M'}h$. (Here we use the following general fact. Let $X$ be a connected CW complex and let $Y$ be a connected subcomplex. Suppose $f, f' \colon X \to K(G,1)$ be maps such that $f|_Y = f'|_Y$ and $f_* = f'_* \colon \pi_1 X \to \pi_1(K(G,1)) = G$. Then $f$ and $f'$ are homotopic through maps restricting to $f|_Y \colon Y \to K(G,1)$.) $\square$

## 4.5 Mapping cylinder

Let $h \colon (M, \rho_M, \phi_M) \overset{\cong}{\to} (M', \rho_{M'}, \phi_{M'})$ be a $G$-diffeomorphism of $(\Sigma, \rho_\Sigma)$-bordered 3–manifolds.

As before, let $C_M = (C_M, \rho_{C_M}, \phi_{C_M})$ denote the reduced cylinder over $M$, which is a cobordism from $M$ to itself.

A *mapping cylinder* associated to $h$ is a cobordism

$$C_h = (C_M, \rho_{C_h}, \phi_{C_h}) \colon (M, \rho_M, \phi_M) \to (M', \rho_{M'}, \phi_{M'})$$

defined as follows. The map

$$\rho_{C_h} \colon C_M \to K(G,1),$$

is induced by a homotopy

$$\rho_{\tilde{C}_h} \colon M \times [0,1] \to K(G,1)$$

realizing (4.4.1). The map $\rho_{C_h}$ is well defined since

$$\rho_{\tilde{C}_h}(x,0) = \rho_M(x), \quad \rho_{\tilde{C}_h}(x,1) = \rho_{M'}(x), \quad \rho_{\tilde{C}_h}(y,t) = \rho_M(y)$$

for $x \in M$, $y \in \partial M$, $t \in [0,1]$. The map

$$\phi_{C_h} \colon M \cup_\Sigma (-M') \overset{\cong}{\to} \partial C_h$$

is obtained by gluing two diffeomorphisms

$$M \overset{\cong}{\to} M \times \{0\}, \quad \text{and} \quad M' \xrightarrow[\cong]{h^{-1}} M \overset{\cong}{\to} M \times \{1\}.$$

By the property of $K(G,1)$, it follows that $C_h$ defines a unique morphism from $M$ to $M'$ in $\mathcal{C}$.

## 4.6 Trace map

Let $W = (W, \rho_W, \phi_W) \colon M \to M$ be an endomorphism of $M = (M, \rho_M, \phi_M) \in \mathrm{Ob}(\mathcal{C})$. Recall that $\phi_W \colon M \cup_\Sigma (-M) \overset{\cong}{\to} \partial W$.

Let $\hat{W}$ denote the closed 4–manifold obtained from $W$ by identifying $\phi_W(M) \subset \partial W$ and $\phi_W(-M) \subset \partial W$ by the diffeomorphism $(\phi_W|_{-M}) \circ (\phi_W|_M)^{-1}$. The map $\rho_W \colon W \to K(G,1)$ induces a map

$$\rho_{\hat{W}} \colon \hat{W} \to K(G,1).$$

Set

$$\mathrm{tr}(W) = [\hat{W}, \rho_{\hat{W}}] \in \Omega_4(G).$$

If $W\colon M \to M$ and $W'\colon M \to M$ are cobordant, then $\hat{W}$ and $\hat{W}'$ are cobordant. Hence we have a function

$$\mathrm{tr}\colon \ \mathrm{End}_{\mathcal{C}}(M) \to \Omega_4(G)$$

For two cobordisms $M \xrightarrow{W} M' \xrightarrow{W'} M$ in $\mathcal{C}$, we have the *trace identity*

$$\mathrm{tr}(W' \circ W) = \mathrm{tr}(W \circ W'). \tag{4.6.1}$$

*Remark* 4.2. The function $\mathrm{tr}\colon \ \mathrm{End}_{\mathcal{C}}(M) \to \Omega_4(G)$ is a group homomorphism. We do not need this fact in the rest of this paper.

## 4.7 Functor induced by a 3-cobordism

Let $\Sigma = (\Sigma, \rho_\Sigma)$ and $\Sigma' = (\Sigma', \rho_{\Sigma'})$ be two $G$-surfaces, and let $M_0 = (M_0, \rho_{M_0}, \phi_{M_0})$ be a cobordism between $\Sigma'$ and $\Sigma$, i.e., $(M_0, \rho_{M_0}, \phi_{M_0})$ is a $(\Sigma' \sqcup (-\Sigma))$-bordered $G$-3–manifold. Then we have a functor

$$F_{M_0}\colon \ \mathcal{C}_\Sigma \to \mathcal{C}_{\Sigma'}$$

defined as follows.

For an object $M = (M, \rho_M, \phi_M) \in \mathrm{Ob}(\mathcal{C}_\Sigma)$, define

$$F_{M_0}((M, \rho_M, \phi_M)) = (F_{M_0}(M), \rho_{F_{M_0}(M)}, \phi_{F_{M_0}(M)}),$$

where

$$\begin{aligned} F_{M_0}(M) &= M \cup_\Sigma M_0, \\ \rho_{F_{M_0}(M)} &= \rho_M \cup \rho_{M_0}, \\ \phi_{F_{M_0}(M)} &= \phi_{M_0}|_{\Sigma'}. \end{aligned}$$

In the following we set $M'' = M \cup_\Sigma M_0$ to simplify the notations. For a morphism

$$(W, \rho_W, \phi_W)\colon \ (M, \rho_M, \phi_M) \to (M', \rho_{M'}, \phi_{M'})$$

in $\mathcal{C}_\Sigma$, set

$$F_{M_0}((W, \rho_W, \phi_W)) = (F_{M_0}(W), \rho_{F_{M_0}(W)}, \phi_{F_{M_0}(W)}).$$

Here

$$F_{M_0}(W) = C_{M''} \cup_M W$$

is obtained by gluing $C_{M''}$ and $W$ along $M$ using the maps

$$M \overset{\phi_W|_M}{\hookrightarrow} \partial W \quad \text{and} \quad M \overset{\cong}{\to} M \times \{0\} \big(\subset M'' \times \{0\} \subset \partial C_{M''}\big).$$

We set

$$\rho_{F_{M_0}(W)} = \rho_W \cup \rho_{C_{M''}} \colon\ F_{M_0}(W) \to K(G,1).$$

The map

$$\phi_{F_{M_0}(W)} \colon\ (M \cup_\Sigma M_0) \cup_{\Sigma'} (-(M' \cup_\Sigma M_0)) \overset{\cong}{\to} \partial(F_{M_0}(W))$$

is defined in an obvious way. It is not difficult to check that $F_{M_0}$ is a well-defined functor.

**Lemma 4.3.** *Let $\Sigma$, $\Sigma'$, $M_0$ be as above. For an endomorphism $W\colon\ M \to M$ in $\mathcal{C}_\Sigma$, we have*

$$\operatorname{tr}(F_{M_0}(W)) = \operatorname{tr}(W) \in \Omega_4(G).$$

*Proof.* Set $W' = F_{M_0}(W)$, and let $\hat{W}'$ be the closed 4–manifold associated to $W'$ as defined in Section 4.6. Consider the cylinder $X = \hat{W}' \times [0,1]$, which is a 5-cobordism between $\hat{W}'$ and itself. Let $\sim$ be the equivalence relation on $\hat{W}' \times \{0\} \subset \partial X$ by

$$((x,t),0) \sim ((x,t'),0)$$

for $x \in M_0$ and $t, t' \in [0,1]$. The 5–manifold $X/\sim$ is a cobordism between $\hat{W}'$ and $\hat{W}$, on which one can construct a structure of a cobordism of closed $G$-4–manifolds in a natural way. Hence we have the result. □

## 4.8 Restatement of Theorem 3.3

As in Sections 2 and 3, let $M$ be a compact, connected, oriented 3–manifold with $\partial M \neq \emptyset$. Let $N$ be a normal subgroup in $\pi_1 M$ and set $G = \pi_1 M/N$. Let $q\colon\ \pi_1 M \to G$ be the projection.

Let $\rho_M\colon\ M \to K(G,1)$ be the composite of the natural maps

$$M \longrightarrow K(\pi_1 M, 1) \overset{K(q,1)}{\longrightarrow} K(G,1).$$

Set $\rho_{\partial M} = \rho_M|_{\partial M}\colon\ \partial M \to K(G,1)$. Note that $M = (M, \rho_M, \mathrm{id}_{\partial M})$ is a $(\partial M, \rho_{\partial M})$-bordered 3–manifold. In the following, we work in the groupoid $\mathcal{C} = \mathcal{C}_{(\partial M, \rho_{\partial M})}$.

Let $L$ be an $N$-link in $M$. Recall that

$$W_L = (M \times [0,1]) \cup (\text{2-handles attached along } L \times \{1\}).$$

By abuse of notation, let $W_L$ denote the 4–manifold obtained from $W_L$ by "reducing $\partial M \times [0,1]$" by the equivalence relation $(x,t) \sim (x,t')$, $x \in \partial M$, $t, t \in [0,1]$. One can identify $W_L$ with

$$C_M \cup (\text{2-handles attached along } L \times \{1\}).$$

The map $\rho_M\colon M \to K(G,1)$ extends to

$$\rho_{W_L}\colon W_L \to K(G,1),$$

which is unique up to homotopy through extensions of $\rho_M$. Set

$$\rho_{M_L} = \rho_{W_L}|_{M_L}\colon M_L \to K(G,1).$$

Then $W_L$ is a cobordism between $(\partial M, \rho_{\partial M})$-bordered 3–manifolds $(M, \rho_M)$ and $(M_L, \rho_{M_L})$.

Let $L'$ be another $N$-link in $M$. If there is a $G$-diffeomorphism

$$h\colon (M_L, \rho_{M_L}, \phi_{M_L}) \overset{\cong}{\to} (M_{L'}, \rho_{M_{L'}}, \phi_{M_{L'}}),$$

then we have

$$\eta_G(M, L, L', h) = \theta_4(\mathrm{tr}([W_{L'}]^{-1} \circ [C_h] \circ [W_L])), \tag{4.8.1}$$

where $C_h\colon M_L \to M_{L'}$ is the mapping cylinder of $h$.

Now, we can restate Theorem 3.3 as follows.

**Theorem 4.4.** *Let $M$, $N$, $G$ be as above. Let $L$ and $L'$ be $N$-links in $M$. Then the following conditions are equivalent.*

*(i) $L$ and $L'$ are $\delta(N)$-equivalent.*

*(ii) There is a $G$-diffeomorphism $h\colon M_L \to M_{L'}$ such that*

$$\theta_4(\mathrm{tr}([W_{L'}]^{-1} \circ [C_h] \circ [W_L])) = 0. \tag{4.8.2}$$

# 5 Framed link realization of homology classes

In this section, we eliminate the identity (4.8.2) in Theorem 4.4 by introducing new moves on framed links.

## 5.1 Framed link realization of $\alpha \in H_4(G)$

A framed link $L$ in a $G$-3–manifold $(M, \rho_M)$ is said to be *$G$-trivial* if $(\rho_M)_*(N_L) = \{1\}$.

**Theorem 5.1.** *Let $G$ be a group, and let $\alpha \in H_4(G)$. Then there are*

- *a $G$-3–manifold $(V, \rho_V)$ with $V$ a handlebody,*
- *a $G$-trivial framed link $K$ in $(V, \rho_V)$,*
- *a diffeomorphism $h_V\colon V_K \overset{\cong}{\to} V$ yielding a mapping cylinder $C_{h_V}\colon (V_K, \rho_{V_K}) \to (V, \rho_V)$ in $\mathcal{C}_{\partial V}$,*

*such that we have*

$$\theta_4(\mathrm{tr}(C_{h_V} \circ W_K^V)) = \alpha. \tag{5.1.1}$$

We call $((V, \rho_V), K, h_V)$ a *framed link realization* of $\alpha$.

*Proof.* Since $\theta_4 \colon \Omega_4(G) \to H_4(G)$ is surjective, $\alpha \in H_4(G)$ is represented by a closed, connected, oriented $G$-4–manifold $(U, \rho_U)$. Thus we have $(\rho_U)_*([U]) = \alpha$, where $[U] \in H_4(U)$ is the fundamental class of $U$.

Suppose that $\pi_1(U)$ is generated by $r(\geq 0)$ elements. Let $V$ denote the 3-dimensional handlebody of genus $r$. Take an embedding $g \colon V \hookrightarrow U$ such that $g_* \colon \pi_1 V \to \pi_1 U$ is surjective. Set $\rho_V = \rho_U g \colon V \to K(G,1)$. Then we have a $(\partial V, \rho_{\partial V})$-bordered 3–manifold $(V, \rho_V, \phi_V)$ in an obvious way.

Let $E$ denote the 4–manifold obtained from $U$ by cutting along the 3-submanifold $g(V)$. We regard $E$ as a cobordism from $V$ to itself. Let $\phi_E \colon V \cup_{\partial V} (-V) \xrightarrow{\cong} \partial E$ be the boundary parameterization. Let $\rho_E \colon E \to K(G,1)$ be the composite of $\rho_U$ with the canonical map $E \to U$. Then we have a cobordism

$$E = (E, \rho_E, \phi_E) \colon (V, \rho_V, \phi_V) \to (V, \rho_V, \phi_V),$$

which represents an endomorphism $E \colon V \to V$ in the category $\mathcal{C}_{\partial V}$. By construction, $\mathrm{tr}(E)$ is cobordant to $(U, \rho_U)$, hence

$$\theta_4(\mathrm{tr}(E)) = \alpha. \tag{5.1.2}$$

Take a handle decomposition of $E$

$$E \cong C_V \cup (\text{1-handles}) \cup (\text{2-handles}) \cup (\text{3-handles}), \tag{5.1.3}$$

where $C_V$ is the reduced cylinder of $V$. We will construct a new cobordism $E' \colon V \to V$ cobordant to $E$ such that $E'$ has a handle decomposition with only 2-handles, by handle-trading as follows.

Suppose that there is a 1-handle $D^3 \times [0,1]$ in the handle decomposition (5.1.3). Let $\gamma = \{0\} \times [0,1] \subset D^3 \times [0,1]$ be the core of the 1-handle. Since $g_*$ is surjective, it follows that there is a path $\gamma'$ in $V \times \{1\} \subset \partial C_V$ such that $\partial\gamma' = \partial\gamma$ and the union $\gamma'' := \gamma \cup \gamma'$ is null-homotopic in $E$. Surgery on $E$ along $\gamma''$ (with any of the two possible framings) gives a 4–manifold $E_{\gamma''}$ cobordant to $E$. Since $\gamma''$ is null-homotopic in $E$, the map $\rho_E \colon E \to K(G,1)$ extends to a map $\rho_{X^E_{\gamma''}} \colon X^E_{\gamma''} \to K(G,1)$, where

$$X^E_{\gamma''} = (C_E \times [0,1]) \cup (\text{2-handle attached along } \gamma'' \times \{1\})$$

is the cobordism between $E$ and $E_{\gamma''}$ associated with the surgery along $\gamma''$. Thus $(E, \rho_E)$ is bordant over $K(G,1)$ to $E_{\gamma''}$. The manifold $E_{\gamma''}$ admits a handle decomposition with the number of 1-handles less by 1 than (5.1.3). By induction, we can trade all 1-handles, and all 3-handles by duality, to obtain a desired cobordism $(E', \rho_{E'}, \phi_{E'})$ between $(V, \rho_V, \phi_V)$ to itself.

Since the cobordism $E'$ has only 2-handles, it follows that the cobordism $E'$ is equivalent to the composite $C_{h_V} \circ W_K^V$, where $K$ is a $G$-trivial framed link in $V$, and $C_{h_V}$ is a mapping cylinder of a $G$-diffeomorphism $h_V\colon V_K \xrightarrow{\cong} V$. It follows that

$$\theta_4(\mathrm{tr}(C_{h_V} \circ W_K^V)) = \theta_4(\mathrm{tr}(E')) = \theta_4(\mathrm{tr}(E)) = \alpha.$$

□

## 5.2 $\alpha$-moves for $\alpha \in H_4(G)$

Let $M$ be a compact, connected, oriented 3–manifold and let $N$ be a normal subgroup of $\pi_1 M$. Set $G = \pi_1 M/N$. We have a $G$-surface $(\partial M, \rho_{\partial M})$ and a $(\partial M, \rho_{\partial M})$-bordered 3–manifold $(M, \rho_M, \phi_M)$.

Let $\alpha \in H_4(G)$, and let $((V, \rho_V), K, h_V)$ be a framed link realization of $\alpha$.

We say that an $N$-link $L'$ in $M$ is obtained from an $N$-link $L$ in $M$ by a $((V, \rho_V), K, h_V)$*-move* if there is an orientation-preserving embedding $f\colon V \hookrightarrow M \setminus L$ such that $\rho_M f \simeq \rho_V\colon V \to K(G,1)$ (rel $\partial V$) and $L'$ is isotopic to $L \cup f(K)$ in $M$. This move preserves the diffeomorphism class of results of surgery. Indeed, there is a diffeomorphism

$$h\colon M_{L\cup f(K)} \xrightarrow{\cong} M_L$$

obtained by gluing $h_V\colon V_K \to V$ and $\mathrm{id}_{M\setminus \mathrm{int}\, f(V)}$. The diffeomorphism $h$ determines a mapping cylinder $C_h$.

**Proposition 5.2.** *In the above situation, we have*

$$\eta_G(M, L \cup f(K), L, h) = \theta_4(\mathrm{tr}(W_L^{-1} \circ C_h \circ W_{L\cup f(K)})) = \alpha \in H_4(G).$$

*Proof.* Note that $W_{L\cup f(K)}\colon M \to M_{L\cup f(K)}$ is cobordant to the composite

$$M \xrightarrow{W_L} M_L \xrightarrow{W_{f(K)}} M_{L\cup f(K)},$$

where

$$W_{f(K)} = W_{f(K)}^{M_L} = (M_L \times [0,1]) \cup (\text{2-handles attached along } f(K) \times \{1\})$$

and we identify $M_{L\cup f(K)}$ with $(M_L)_{f(K)}$. Hence we have

$$\begin{aligned}
\theta_4(\mathrm{tr}(W_L^{-1} \circ C_h \circ W_{L\cup f(K)})) &= \theta_4(\mathrm{tr}(W_L^{-1} \circ C_h \circ W_{f(K)} \circ W_L)) \\
&= \theta_4(\mathrm{tr}(C_h \circ W_{f(K)} \circ W_L \circ W_L^{-1})) && \text{by (4.6.1)} \\
&= \theta_4(\mathrm{tr}(C_h \circ W_{f(K)})) \\
&= \theta_4(\mathrm{tr}(F_{M_L\setminus \mathrm{int}\, f(V)}(C_{h_V} \circ W_K^V))) \\
&= \theta_4(\mathrm{tr}(C_{h_V} \circ W_K^V)) && \text{by Lemma 4.3} \\
&= \alpha.
\end{aligned}$$

□

The following fact follows immediately from Theorem 4.4 and Proposition 5.2.

**Proposition 5.3.** *A framed link realization* $((V, \rho_V), K, h_V)$ *of* $\alpha \in H_4(G)$ *is unique up to* $\delta(N)$*-equivalence in the following sense. Let* $((V', \rho_{V'}), K', h_{V'})$ *be another framed link realization of* $\alpha$*. Then for any* $N$*-link* $L$ *and* $L'$ *in* $M$ *the following conditions are equivalent*

*(i)* $L'$ *is* $\delta(N)$*-equivalent to an* $N$*-link* $L \cup f(K)$ *in* $M$ *obtained by a* $((V, \rho_V), K, h_V)$*-surgery.*

*(ii)* $L'$ *is* $\delta(N)$*-equivalent to an* $N$*-link* $L \cup f(K')$ *in* $M$ *obtained by a* $((V', \rho_{V'}), K', h_{V'})$*-surgery.*

We say that an $N$-link $L'$ in $M$ is obtained from another $N$-link $L$ in $M$ by an *$\alpha$-move* if $L'$ is obtained from $L$ by a $((V, \rho_V), K, h_V)$-move for some framed link realization $((V, \rho_V), K, h_V)$ of $\alpha$.

We have the following characterization of $G$-diffeomorphism of results of surgeries.

**Theorem 5.4.** *Let* $M$ *be a compact, connected, oriented* 3*–manifold, with non-empty boundary. Let* $N$ *be a normal subgroup of* $\pi_1 M$*, and set* $G = \pi_1 M/N$*. Let* $\{\alpha_i\}_{i \in I}$*, with* $I$ *an index set, be a set of generators of the group* $H_4(G)$*. Let* $L$ *and* $L'$ *be* $N$*-links in* $M$*. Then the following conditions are equivalent.*

*(i)* $(M_L, \rho_{M_L}, \phi_{M_L})$ *and* $(M_{L'}, \rho_{M_{L'}}, \phi_{M_{L'}})$ *are* $G$*-diffeomorphic.*

*(ii)* $L$ *and* $L'$ *are related by a sequence of* $\alpha_i$*-moves for* $i \in I$ *and* $\delta(N)$*-equivalence.*

# 6 IHX-moves

In this section, we define $Y_2$-claspers in a 3–manifold, which are special kind of claspers introduced in [8, 9, 10] and used in the theory of finite type invariants of links and 3–manifolds [2, 17]. To each clasper a framed link is associated on which one can perform surgery. We define an IHX-move on the framed links associated to the disjoint union of $Y_2$-claspers. This move preserves the result of surgery up to diffeomorphism. An IHX-move is closely related to the IHX-relation in the theory of finite type invariants. This move is related to a handle decomposition of the 4-torus $T^4$.

## 6.1 $Y_2$-Claspers

Let $M$ be a compact, oriented, connected 3–manifold. We define $Y_2$-claspers in $M$, which is a special kind of tree claspers [8, 9, 10].

A *$Y_2$-clasper* in $M$ is a subsurface embedded in the interior of $M$ which is decomposed into four annuli, two disks and five bands as depicted in Figure 6 (a). We usually depict a $Y_2$-clasper as a framed graph as in Figure 6 (b) using

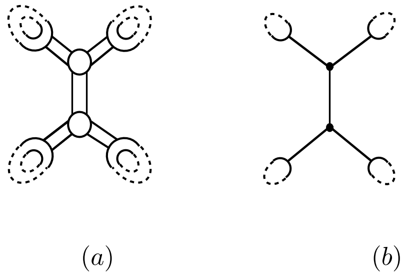


Figure 6: (a) $Y_2$-clasper $T$. (b) Drawing of $T$.

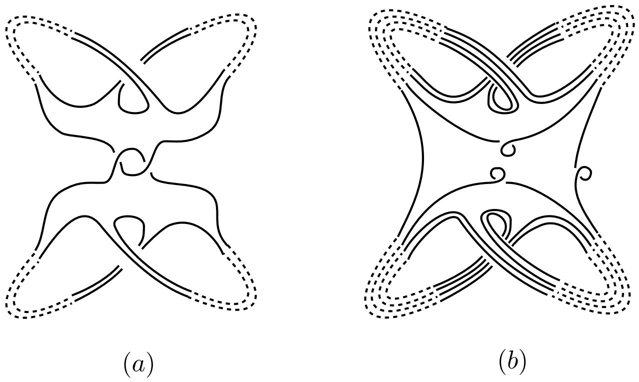


Figure 7: (a) The framed link $L_T = L_{T,1} \cup L_{T,2}$ associated to the $Y_2$-clasper $T$. (b) Another framed link $L_T^{\mathrm{adm}}$ associated to $T$.

the blackboard framing convention.

We associate to a $Y_2$-clasper $T$ in $M$ a 2-component framed link $L_T$ in the small regular neighborhood $N(T)$ of $T$ in $M$ as depicted in Figure 7 (a). Note that the framed link $L_T$ is $\mathbb{Z}$-null-homologous in $N(T)$, hence in $M$. Surgery along the $Y_2$-clasper $T$ is defined to be surgery along the associated framed link $L_T$.

Figure 7 (b) shows another framed link $L_T^{\mathrm{adm}}$ associated to $T$, called the *associated admissible framed link* of $T$, which is used in Section 9.1.

**Lemma 6.1.** *The framed links $L_T^{\mathrm{adm}}$ and $L_T$ in $N(T)$ are $\delta$-equivalent.*

*Proof.* By using one stabilization and two handle-slides, we obtain from $L_T$ the framed link $L'_T$ depicted in Figure 8. Then, by handle-sliding the middle component over the other two components in $L'_T$, we obtain $L_T^{\mathrm{adm}}$. $\square$

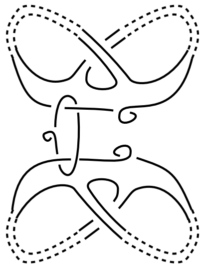


Figure 8: The framed link $L'_T$.

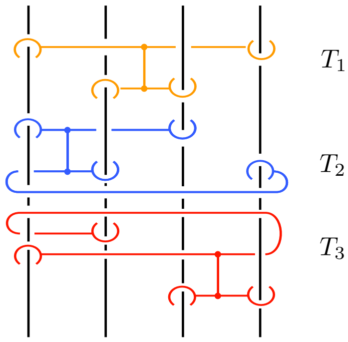



Figure 9: $T_{IHX} \subset V_4$

## 6.2 IHX-claspers and IHX-links

Let $\gamma = \gamma_1 \cup \cdots \cup \gamma_4$ be a trivial string link in the cylinder $D^2 \times [0,1]$, i.e., $\gamma$ is a proper 1-submanifold of $D^2 \times [0,1]$ of the form

$$(4 \text{ points in } \operatorname{int} D^2) \times [0,1].$$

Let $N(\gamma) \subset D^2 \times [0,1]$ be a small tubular neighborhood of $\gamma$ in $D^2 \times [0,1]$, and set

$$V_4 = \overline{(D^2 \times [0,1]) \setminus N(\gamma)}.$$

Let $T_{IHX} = T_1 \cup T_2 \cup T_3 \subset V_4$ be the disjoint union of three $Y_2$-claspers $T_1, T_2, T_3$ as depicted in Figure 9. $T_{IHX}$ is called the *IHX-clasper*.

**Theorem 6.2.** *Surgery along $T_{IHX}$ preserves the manifold $V_4$. More precisely, There is a diffeomorphism*

$$h_V \colon (V_4)_{T_{IHX}} \xrightarrow{\cong} V_4 \tag{6.2.1}$$

*restricting to $\mathrm{id}_{\partial V_4}$. (Note that such a diffeomorphism is unique up to isotopy fixing $\partial V_4$.)*

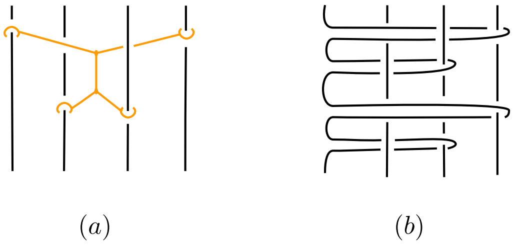


Figure 10: (a) The $Y_2$-clasper $T_1$ and the trivial string link $\gamma$. (b) The pure braid $\beta_1$.

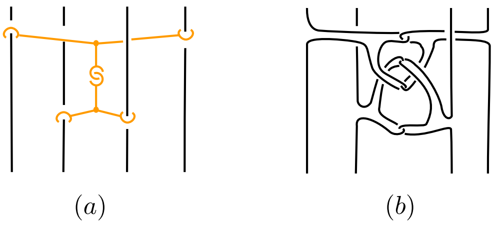


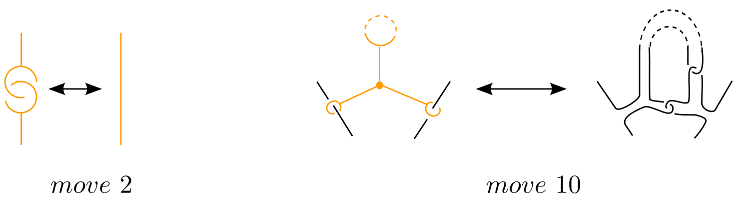
Figure 11: (a) $T_1$ after move 2. (b) $\gamma$ after surgery along $T_1$.

Theorem 6.2 is closely related to the IHX relation in the theory of finite type invariants. A similar result, with a different configuration of $Y_2$-claspers, has been obtained in [5].

To prove Theorem 6.2, we need the following.

**Lemma 6.3.** *Let $T_1 \subset V_4$ be the first component of $T_{IHX}$, see Figure 10 (a). By surgery along $T_1$, we obtain from $\gamma$ a pure braid $\beta_1 := \gamma_{T_1}$ as depicted in Figure 10 (b). (Here string links are considered to be framed.)*

*Proof.* By clasper calculus (see [10]) we can transform $(\gamma, T_1)$ into $(\beta_1, \emptyset)$ as follows. Consider the two clasper operations, which do not change the isotopy class of the result of surgery, *move 2* and *move 10* given in [10].



First, apply move 2 to the clasper $T_1$ as shown in Figure 11 (a). Then, apply move 10 twice to obtain Figure 11 (b). This string link is isotopic to $\beta_1$. $\square$


Figure 12: Braids $\alpha$ and $\alpha^{-1}$

*Proof of Theorem 6.2.* First, we see that the pairs $(\gamma, T_2)$ and $(\gamma, T_3)$ are conjugate with $(\gamma, T_1)$ as follows. Let $\alpha^{\pm 1}$ be the braids depicted in Figure 12. Then we have

$$(\gamma, T_2) \cong \alpha^2(\gamma, T_1)\alpha^{-2},$$
$$(\gamma, T_3) \cong \alpha(\gamma, T_1)\alpha^{-1}.$$

Here the composition $\beta\beta'$ of two tangles possibly with claspers is obtained from stacking $\beta$ on the top of $\beta'$.

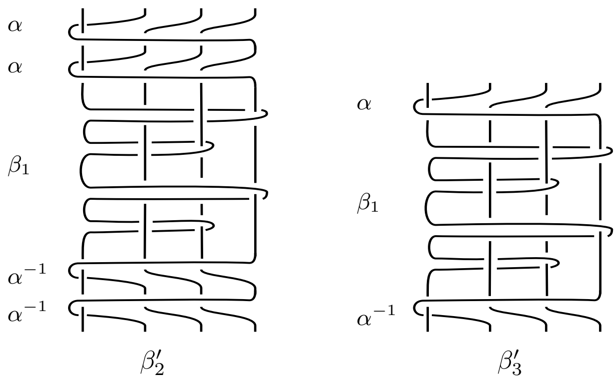
Then, the result $\gamma_{T_2}$ from $\gamma$ by surgery along $T_2$ is the conjugate

$$\gamma_{T_2} \cong \alpha^2\gamma_{T_1}\alpha^{-2} \cong \alpha^2\beta_1\alpha^{-2} =: \beta_2',$$

where we used $\gamma_{T_1} \cong \beta_1$ (Lemma 6.3). Similarly, we have

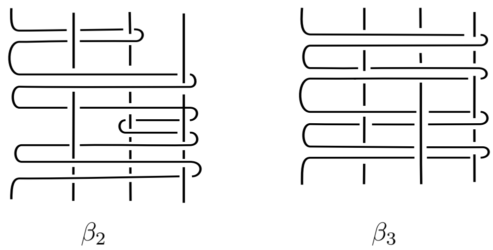
$$\gamma_{T_3} \cong \alpha\gamma_{T_1}\alpha^{-1} \cong \alpha\beta_1\alpha^{-1} =: \beta_3'.$$

These are pure braids depicted below:




.

By isotopy we obtain the braids $\beta_2$ and $\beta_3$

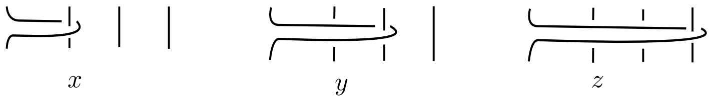


Now, one can check that the composition $\beta_1 \circ \beta_2 \circ \beta_3$ is isotopic to the trivial string link $\gamma$. Thus, surgery along $T_{IHX}$ preserves $V_4$. $\square$

*Remark* 6.4. Theorem 6.2 may be regarded as a topological version of the Witt-Hall identity

$$[z, [y^{-1}, x]]^{y^{-1}} \cdot [y, [x^{-1}, z]]^{x^{-1}} \cdot [x, [z^{-1}, y]]^{z^{-1}} = 1$$

in a free group inside the pure braid group, where we define $x, y, z$ by




and $[x, y] = xyx^{-1}y^{-1}$ is the commutator.

## 6.3 IHX-moves

Let $L$ be a framed link in a 3–manifold $M$. Let $f\colon\ V_4 \hookrightarrow M \setminus L$ be an orientation-preserving embedding. Then the framed links $L$ and $L \cup f(L_{IHX})$ are said to be related by an *IHX-move*.

An IHX-move preserves the result of surgery. More precisely, there is a diffeomorphism

$$h\colon\ M_{f(L_{IHX})} \to M$$

restricting to $\mathrm{id}_{M \setminus \mathrm{int}\, f(V)}$, which is unique up to isotopy through such diffeomorphisms. Indeed, the diffeomorphism $h$ is obtained by gluing the composite

$$f(V)_{f(L_{IHX})} \cong V_{L_{IHX}} \xrightarrow[\cong]{h_V} V \cong f(V)$$

and $\mathrm{id}_{M \setminus \mathrm{int}\, f(V)}$.

Note that if $L'$ is obtained from a $\mathbb{Z}$-(resp. $\mathbb{Q}$-)null-homologous framed link $L$ by an IHX-move, then $L'$ is again $\mathbb{Z}$-(resp. $\mathbb{Q}$-)null-homologous.

## 6.4 Admissible IHX moves

The definition of an *admissible IHX-move* on a framed link is the same as that of an IHX-move except that we use $L^{\mathrm{adm}}_{IHX}$ instead of $L_{IHX}$. Clearly, if an admissible IHX-move is applied to an admissible framed link, then the result is again admissible.

The following lemma immediately follows from Lemma 6.1.

**Lemma 6.5.** *An IHX-move can be realized by admissible IHX-moves, stabilizations, and handle-slides. Conversely, an admissible IHX-move can be realized by IHX-moves, stabilizations, and handle-slides.*

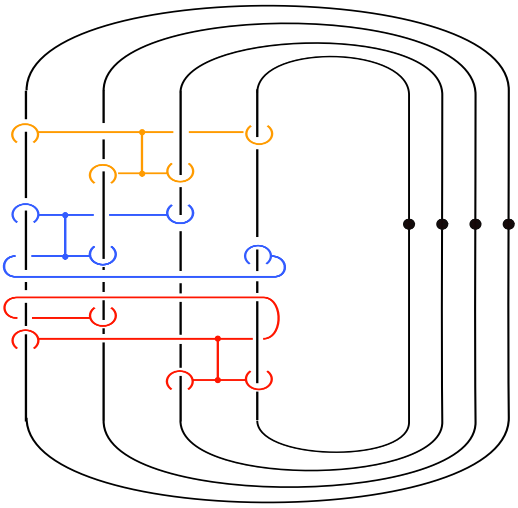


Figure 13: Handle decomposition for the 4–manifold $W$

# 7 A handle decomposition of $T^4$

In this section, we construct a new handle decomposition of the 4-torus $T^4$ involving the IHX-link.

Consider the framed link with dotted circles obtained from the IHX-link $L_{IHX} \subset V_4 \subset D^2 \times [0,1]$ as follows. We embed $D^2 \times [0,1]$ into $S^3$, close the trivial string link $\gamma$ in a natural way to obtain an unlink $J = J_1 \cup \cdots \cup J_4$, and put a dot on each component of $J$. Here, each $Y_2$-clasper $T_i$ of $T_{IHX} = T_1 \cup T_2 \cup T_3$ is regarded as its associated framed link which we denote by $K_i \cup K_i'$. The framed link

$$(J_1 \cup \cdots \cup J_4) \cup (K_1 \cup K_1' \cup K_2 \cup K_2' \cup K_3 \cup K_3') \subset S^3$$

gives a handlebody $W^{(2)}$ consisting of one 0-handle $W^{(0)} = B^4$, four 1-handles $B_1, \ldots, B_4$ corresponding to $J_1, \ldots, J_4$, and six 2-handles $H_1, H_1', H_2, H_2', H_3, H_3'$ corresponding to $K = K_1 \cup K_1' \cup K_2 \cup K_2' \cup K_3 \cup K_3'$. We set

$$W^{(1)} = W^{(0)} \cup B_1 \cup \cdots \cup B_4$$
$$W^{(2)} = W^{(1)} \cup H_1 \cup H_1' \cup \cdots \cup H_3 \cup H_3'$$

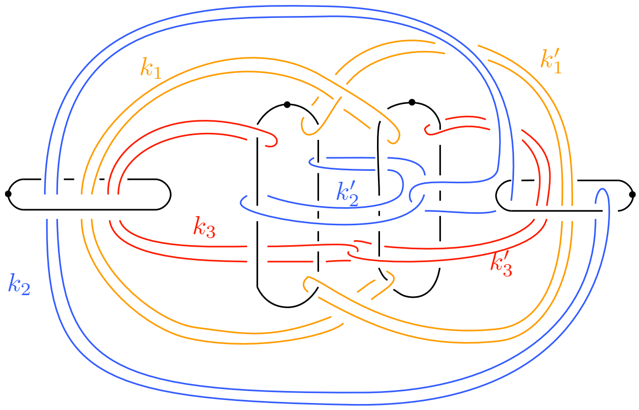
Since surgery along $K$ preserves the result of surgery, we have

$$\partial W^{(2)} \cong \partial W^{(1)} \cong \sharp^4 (S^2 \times S^1).$$

Hence we can attach four 3-handles and one 4-handle to obtain a oriented, closed 4–manifold $W$.

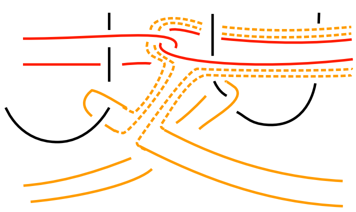
**Theorem 7.1.** *The* 4*–manifold* $W$ *is diffeomorphic to the* 4*-torus* $T^4$*. Thus the framed link obtained from Figure 13 by replacing* $Y_2$*-claspers with the associated framed link presents a handle decomposition of* $T^4$*.*

*Proof.* We start from the following handle decomposition of $T^4$ obtained by Akbulut in [1].

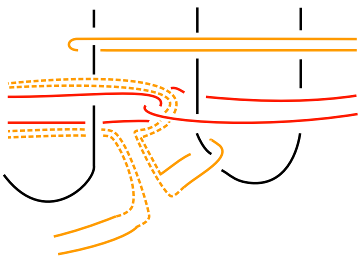



We perform a sequence of handle-slides on the six 2-handles, i.e. on the link $k = k_1 \cup k_1' \cup k_2 \cup k_2' \cup k_3 \cup k_3'$.

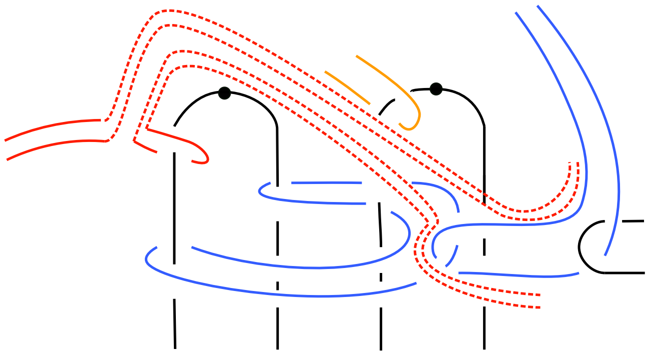
Slide $k_1'$ twice over $k_3'$ as follows.



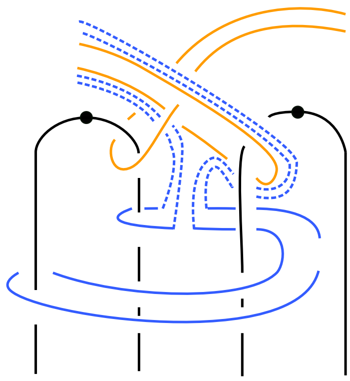
Slide $k_1$ twice over $k_3$ as follows.



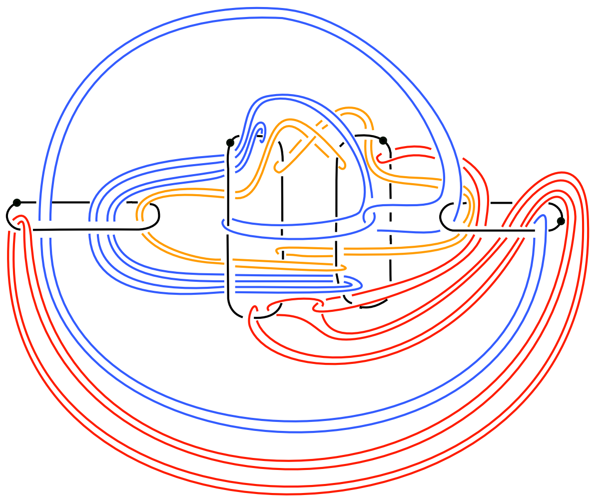
Slide $k_3$ twice over $k_2$ as follows.



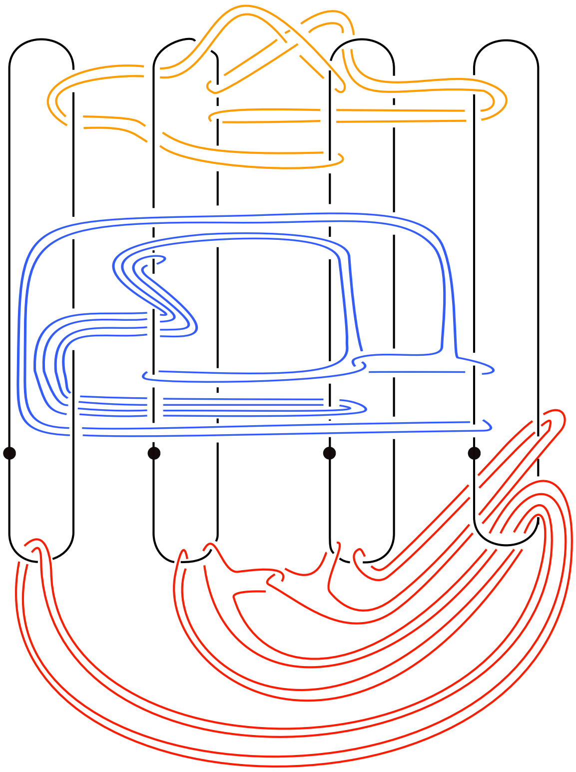
Slide $k_2'$ twice over $k_1'$ as follows.



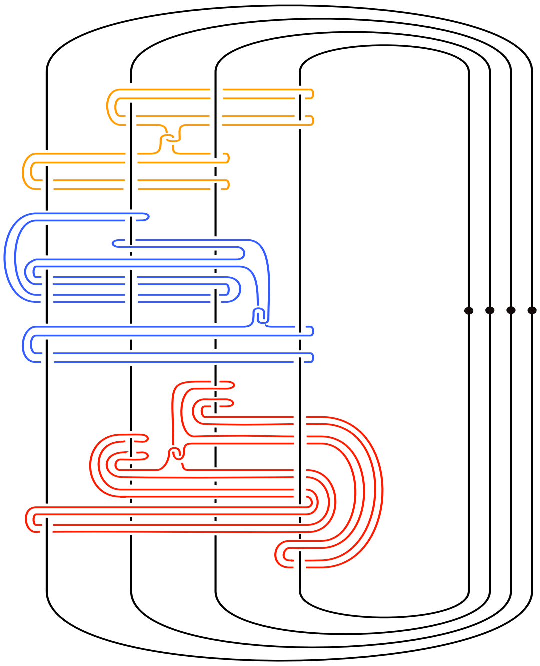
After isotopy, we obtain the following.



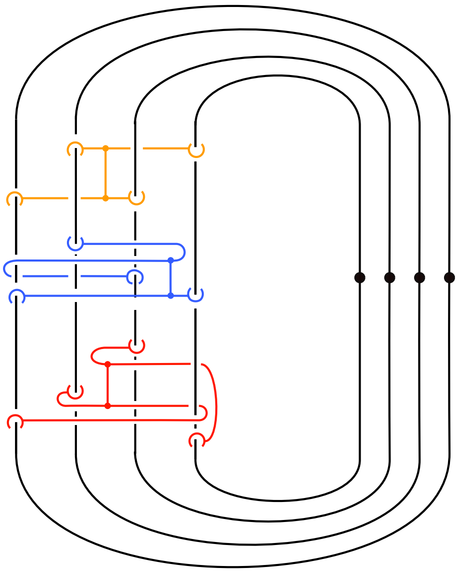
The three 2-component links can be separated as follows



The following shows the result after rearranging the dotted circles.



In clasper calculus this corresponds to the following.



Now, scale down the outermost dotted circle by isotopy passing under the second one until it becomes the second circle. This yields Figure 13. □

# 8 Kirby calculus for $\mathbb{Q}$-null-homologous framed links

Let $M$ be a compact, connected, oriented 3–manifold, and let $P = \{p_1, \dots, p_t\} \subset \partial M$ be as in Section 2. In this section, we consider the case where

$$N = \ker(\pi_1 M \to H_1(M, \mathbb{Q})).$$

The quotient

$$G = \pi_1 M / N \cong H_1 M / (\text{torsion})$$

is a free abelian group. We fix an identification

$$G = \mathbb{Z}^r,$$

where $r = \operatorname{rank}(H_1 M)$.

## 8.1 The homology group $H_4(\mathbb{Z}^r)$

In the following, we often identify $H_1 G = H_1 \mathbb{Z}^r$ with $G = \mathbb{Z}^r$.

As is well-known, the Pontryagin product (see e.g. [13])

$$H_1(\mathbb{Z}^r) \otimes H_1(\mathbb{Z}^r) \otimes H_1(\mathbb{Z}^r) \otimes H_1(\mathbb{Z}^r) \to H_4(\mathbb{Z}^r)$$

induces an isomorphism

$$p\colon \bigwedge^4 H_1(\mathbb{Z}^r) \overset{\cong}{\to} H_4(\mathbb{Z}^r). \tag{8.1.1}$$

Define $y_1, \dots, y_4 \in H_1 T^4$ by

$$y_1 = [S^1 \times \mathrm{pt} \times \mathrm{pt} \times \mathrm{pt}],\ \dots,\ y_4 = [\mathrm{pt} \times \mathrm{pt} \times \mathrm{pt} \times S^1].$$

Then $y_1, \dots, y_4$ generate $H_1 T^4 \cong \mathbb{Z}^4$, We have

$$p(y_1 \wedge \cdots \wedge y_4) = [T^4], \tag{8.1.2}$$

where $[T^4] \in H_4 T^4$ is the fundamental class.

The following lemma follows from the definition of the Pontryagin product.

**Lemma 8.1.** *Let $\rho_{T^4}\colon T^4 \to K(\mathbb{Z}^r, 1)$ be a map. Then we have*

$$(\rho_{T^4})_*([T^4]) = p((\rho_{T^4})_*(y_1) \wedge \cdots \wedge (\rho_{T^4})_*(y_4)) \in H_4(\mathbb{Z}^r).$$

*Proof.* We have

$$\begin{aligned} (\rho_{T^4})_*([T^4]) &= (\rho_{T^4})_*(p(y_1 \wedge \cdots \wedge y_4)) && \text{by (8.1.2)} \\ &= p((\rho_{T^4})_*(y_1) \wedge \cdots \wedge (\rho_{T^4})_*(y_4)) && \text{by naturality of } p. \end{aligned}$$

Here we used the fact that $\rho_{T^4}\colon T^4 \to K(\mathbb{Z}^r, 1) = T^r$ is homotopic to a Lie group homomorphism. $\square$

## 8.2 Effect of an IHX-move in $H_4(\mathbb{Z}^r)$

As in Section 6, let $V = V_4$ be a handlebody of genus 4 obtained from the cylinder $D^2 \times [0,1]$ by removing the interiors of the tubular neighborhood of a trivial 4-component string link $\gamma = \gamma_1 \cup \cdots \cup \gamma_4$. For $i = 1, \ldots, 4$, let $x_i \in H_1 V$ be the meridian to $\gamma_i$.

Suppose that we are given a $\mathbb{Z}^r$–manifold $(M, \rho_M)$ such that $(\rho_M)_* \colon \pi_1 M \to \mathbb{Z}^r$ is surjective.

Let $y_1, \ldots, y_4 \in H_1 M$. Let $f\colon V \hookrightarrow M$ be an orientation-preserving embedding such that $f_*(x_i) = y_i$, $i = 1, \ldots, 4$.

Set $\rho_V = \rho_M f\colon V \to K(\mathbb{Z}^r, 1)$. Then $(V, \rho_V)$ is a $\mathbb{Z}^r$–manifold.

Recall that $L_{IHX}$ denotes the IHX link in $V$. Set $L = f(L_{IHX})$, which is a $\mathbb{Z}$-null-homologous framed link in $M$. The diffeomorphism $h_V \colon V_{L_{IHX}} \overset{\cong}{\to} V$ naturally extends to a diffeomorphism

$$h = h_V \cup \mathrm{id}_{M \setminus \mathrm{int}\, f(V)} \colon M_L \overset{\cong}{\to} M.$$

The following result describes the effect of an IHX-move on the homology class in $H_4(\mathbb{Z}^r)$.

**Proposition 8.2.** *In the above situation, we have*

$$\theta_4(\mathrm{tr}(C_h \circ W_L)) = \pm p(y_1 \wedge \cdots \wedge y_4) \in H_4(\mathbb{Z}^r). \tag{8.2.1}$$

*Proof.* Let $Y = V \cup_\partial (-V) \cong \sharp^4(S^2 \times S^1)$ be the double of $V$, and let $i\colon V \hookrightarrow Y$ be the inclusion. The diffeomorphism $h_V \colon V_L \overset{\cong}{\to} V$ extends to $h_Y = h \cup \mathrm{id}_{-V} \colon Y_L \overset{\cong}{\to} Y$. Set $\rho_Y = \rho_V \cup \rho_{-V} \colon Y \to K(\mathbb{Z}^r, 1)$, where $\rho_{-V} \colon -V \to K(\mathbb{Z}^r, 1)$ is the same as $\rho_V$.

By using Lemma 4.3 twice for inclusions $M \supset V \subset Y$, we have

$$\theta_4(\mathrm{tr}(C_h \circ W_L^M)) = \theta_4(\mathrm{tr}(C_{h_V} \circ W_L^V)) = \theta_4(\mathrm{tr}(C_{h_Y} \circ W_L^Y))$$

in $H_4(\mathbb{Z}^r)$.

In the following, we will show that the closed 4–manifold $\mathrm{tr}(C_{h_Y} \circ W_L^Y)$ is cobordant to $W \cong T^4$ over $K(\mathbb{Z}^r, 1)$, where $W$ is defined in Section 7.

Consider the cylinder $C_Y := Y \times [0,1]$ and define a map $\rho_{C_Y} \colon C_Y \to K(\mathbb{Z}^r, 1)$ as the composite

$$C_Y \xrightarrow{\mathrm{proj}} Y \xrightarrow{\rho_Y} K(\mathbb{Z}^r, 1).$$

The 3–manifold $Y$ is naturally identified with the boundary of the 4-dimensional handlebody

$$Z := B^4 \cup (\text{four 1-handles})$$

We regard $Z$ as a cobordism $Z\colon \emptyset \to Y$ (over $K(\mathbb{Z}^r, 1)$) from the empty 3–manifold $\emptyset$ to $Y$. The orientation-reversal $-Z$ of $Z$ is regarded as a cobordism

$-Z\colon Y \to \emptyset$. Then the cobordism $C_Y\colon Y \to Y$ is cobordant over $K(\mathbb{Z}^r, 1)$ to $Z \circ (-Z)\colon Y \to Y$.

Then we have

$$\begin{aligned}\theta_4(\mathrm{tr}(C_{h_Y} \circ W_L^Y)) =& \theta_4(\mathrm{tr}(C_{h_Y} \circ W_L^Y \circ C_Y))\\ =& \theta_4(\mathrm{tr}(C_{h_Y} \circ W_L^Y \circ Z \circ (-Z)))\\ =& \theta_4(\mathrm{tr}((-Z) \circ C_{h_Y} \circ W_L^Y \circ Z))\\ =& \theta_4(W, \rho_W).\end{aligned}$$

The last identity follows from natural diffeomorphism of closed 4–manifolds

$$g\colon\ (-Z) \circ C_{h_Y} \circ W_L^Y \circ Z \xrightarrow{\cong} W,$$

The map $\rho_W\colon W \to K(\mathbb{Z}^r, 1)$ is the one which extends $\rho_Y\colon Y \to K(\mathbb{Z}^r, 1)$.

By Theorem 7.1, we have a diffeomorphism

$$g'\colon\ W \xrightarrow{\cong} T^4.$$

Define $\rho_{T^4}\colon T^4 \to K(\mathbb{Z}^r, 1)$ as the composite

$$T^4 \underset{\cong}{\xrightarrow{(g')_*^{-1}}} W \xrightarrow{\rho_W} K(\mathbb{Z}^r, 1).$$

Clearly, we have

$$\theta_4(W, \rho_W) = \theta_4(T^4, \rho_{T^4}).$$

Let $j\colon Y \hookrightarrow W$ be the inclusion map. By construction, we see that $(g'gji)_*(x_i) \in H_1T^4$, $i = 1, \dots, 4$, are a set of generators of $H_1T^4 \cong \mathbb{Z}^4$. Hence we have

$$\begin{aligned}\theta_4(T^4, \rho_{T^4}) &= \pm p((\rho_{T^4}g'gji)_*(x_1) \wedge \cdots \wedge (\rho_{T^4}g'gji)_*(x_4))\\ &= \pm p((\rho_V)_*(x_1) \wedge \cdots \wedge (\rho_V)_*(x_4))\\ &= \pm p(y_1 \wedge \cdots \wedge y_4).\end{aligned}$$

The identity (8.2.1) follows from the above identities. □

**Theorem 8.3.** *Let $M$ be a compact, connected, oriented 3–manifold with non-empty boundary. Let $N$ be a normal subgroup of $\pi_1 M$ such that $\pi_1 M/N \cong \mathbb{Z}^r$ with $r \geq 0$. (Here $r$ may or may not be equal to the rank of $H_1 M$.) Let $L$ and $L'$ be two $N$-links. Then the following conditions are equivalent.*

*(i) $(M_L, \rho_{M_L}, \phi_{M_L})$ and $(M_{L'}, \rho_{M_{L'}}, \phi_{M_{L'}})$ are $\mathbb{Z}^r$-diffeomorphic.*

*(ii) $L$ and $L'$ are related by a sequence of IHX-moves and $\delta(N)$-equivalence.*

*Proof.* The result follows from Theorem 5.4 and Proposition 8.2 since the set

$$\{p(z_1 \wedge \cdots \wedge z_4) \in H_4(\mathbb{Z}^r) \mid z_1, \dots, z_4 \in H_1 M\}$$

generates the group $H_4(\mathbb{Z}^r)$. □

*Remark* 8.4. If $r \leq 3$, then we do not need IHX-moves in Theorem 8.3 since $H_4(\mathbb{Z}^r) = 0$.

## 8.3 Proof of Theorem 1.1

Here we consider a special case of Theorem 8.3, where $N$ is the kernel of the map $\pi_1 M \to H_1 M \to H_1(M;\mathbb{Q})$.

In the present situation, a framed link $L$ in $M$ is an $N$-link if and only if it is a $\mathbb{Q}$-null-homologous framed link as defined in Section 1. An $N$-move is the same as a $\mathbb{Q}$-*null-homologous* $K_3$*-move*.

The following result includes Theorem 1.1.

**Theorem 8.5.** *Let $M$ be a compact, connected, oriented 3–manifold with non-empty boundary with* $\operatorname{rank} H_1 M = r \geq 0$*, which we regard as a $(\partial M, \rho_{\partial M})$-bordered $\mathbb{Z}^r$–manifold $(M, \rho_M, \phi_M)$. Let $L$ and $L'$ be $\mathbb{Q}$-null-homologous framed links in $M$. Then the following conditions are equivalent.*

*(i) $(M_L, \rho_{M_L}, \phi_{M_L})$ and $(M_{L'}, \rho_{M_{L'}}, \phi_{M_{L'}})$ are $\mathbb{Z}^r$-diffeomorphic.*

*(ii) $L$ and $L'$ are related by a sequence of stabilizations, handle-slides, $\mathbb{Q}$-null-homologous $K_3$-moves and IHX-moves.*

*(iii) There is a diffeomorphism $h\colon M_L \xrightarrow{\cong} M_{L'}$ restricting to the canonical identification $\partial M_L \cong \partial M_{L'}$ such that the following diagram commutes*

$$\begin{tikzcd} H_1(M_L, P_L; \mathbb{Q}) \arrow[rr, "h_*", "\cong"'] \arrow[dr, "g_L"'] & & H_1(M_{L'}, P_{L'}; \mathbb{Q}) \arrow[dl, "g'_L"] \\ & H_1(M, P; \mathbb{Q}). & \end{tikzcd} \tag{8.3.1}$$

*See Section 1 for the definition of $g_L, g_{L'}$.*

*Proof.* By Theorem 8.3, Conditions (i) and (ii) are equivalent.

By Proposition 4.1 we see that Condition (i) is equivalent to:

(iii') There is a diffeomorphism $h\colon M_L \xrightarrow{\cong} M_{L'}$ restricting to $\partial M_L \cong \partial M_{L'}$ such that the following groupoid diagram commutes:

$$\begin{tikzcd} \Pi(M_L, P_L) \arrow[rr, "h_*", "\cong"'] \arrow[dr, "q_L"'] & & \Pi(M_{L'}, P_{L'}) \arrow[dl, "q_{L'}"] \\ & \Pi(M, P)/N & \end{tikzcd} \tag{8.3.2}$$

where $N = \ker \pi_1 M \to H_1(M;\mathbb{Q})$, and $q_L$, $q_{L'}$ are as defined in (3.1.2).

Then one easily checks that Conditions (iii) and (iii') are equivalent. $\square$

# 9 Admissible framed links in 3–manifolds with free abelian first homology group

In this section, we prove Theorem 1.4.

## 9.1 Admissible IHX-moves

Let $M$ be a compact, connected, oriented 3–manifold.

## 9.2 Reduction of Theorem 1.4

By Theorem 1.2, we see that Theorem 1.4 follows from the following result.

**Proposition 9.1.** *Let $M$ be a compact, connected, oriented 3–manifold with non-empty boundary, such that $H_1M \cong \mathbb{Z}^r$, $r \geq 0$. Let $L$ and $L'$ be two admissible framed links in a compact, connected, oriented 3–manifold $M$. (Here $M$ may be closed and may have $H_1M$ with nontrivial torsion.) Then the following conditions are equivalent.*

*(i) $L$ and $L'$ are related by a sequence of stabilization, band-slides, pair-moves, admissible IHX-moves, and lantern-moves.*

*(ii) $L$ and $L'$ are related by a sequence of stabilization, handle-slides, $\mathbb{Z}$-null-homologous $K_3$-moves and IHX-moves.*

*Proof of Proposition 9.1, (i) implies (ii).* We have seen that stabilizations, band-slides, pair-moves, admissible IHX-moves are realized by a sequence of stabilizations, handle-slides, $\mathbb{Z}$-null-homologous $K_3$-moves and IHX-moves.

We will show that a lantern-move is realized by a sequence of stabilizations, handle-slides and $\mathbb{Z}$-null-homologous $K_3$-moves. Let $K$ and $K'$ be the two framed links in $V_3$ depicted in Figure 5(a) and (b). Figure 14 shows a sequence of stabilizations, handle-slides and $\mathbb{Z}$-null-homologous $K_3$-moves from $K$ to a framed link $\tilde{K}$. Similarly, Figure 15 shows a sequence of stabilizations, handle-slides and $\mathbb{Z}$-null-homologous $K_3$-moves from $K'$ to a framed link $\tilde{K'}$. The links $\tilde{K}$ and $\tilde{K'}$ are isotopic. Thus, there exists a sequence of stabilizations, handle-slides and $\mathbb{Z}$-null-homologous $K_3$-moves from $K$ to $K'$. □

In the rest of this section, we prove that (ii) implies (i).

## 9.3 The category $\mathcal{S}_{M,n}$

In the proof that (ii) implies (i) in Proposition 9.1, we use oriented, ordered framed links. We briefly recall some definitions and results from [11].

An *oriented, ordered framed link* in a 3–manifold $M$ is a framed link $L = L_1 \cup \cdots \cup L_n$ in $M$ such that each component $L_i$ of $L$ is given an orientation, and the set of components of $L$ is given a total ordering. Two oriented, ordered framed links are considered equivalent if there is an ambient isotopy between them preserving the orientations and the orderings.

Following [11], let $\mathcal{L}_{M,n}$, $n \geq 0$, denote the set of equivalence classes of oriented, ordered framed links in $M$. Let $\mathcal{E} = \mathcal{E}_n$ denote the set of symbols

- $\mathcal{P}_{i,j}$ for $i, j \in \{1, \ldots, n\}$, $i \neq j$,
- $\mathcal{Q}_i$ for $i \in \{1, \ldots, n\}$,

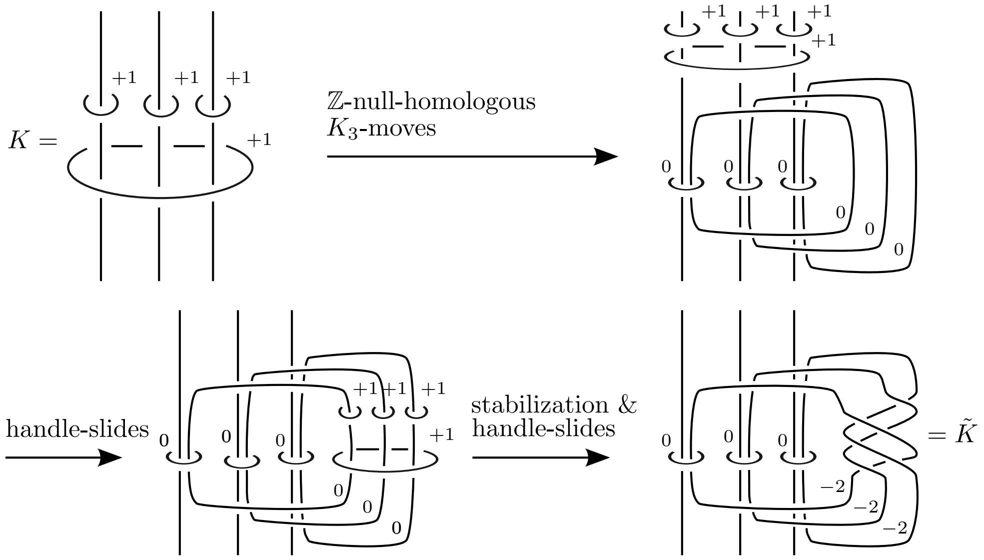



Figure 14: From $K$ to $\tilde{K}$.

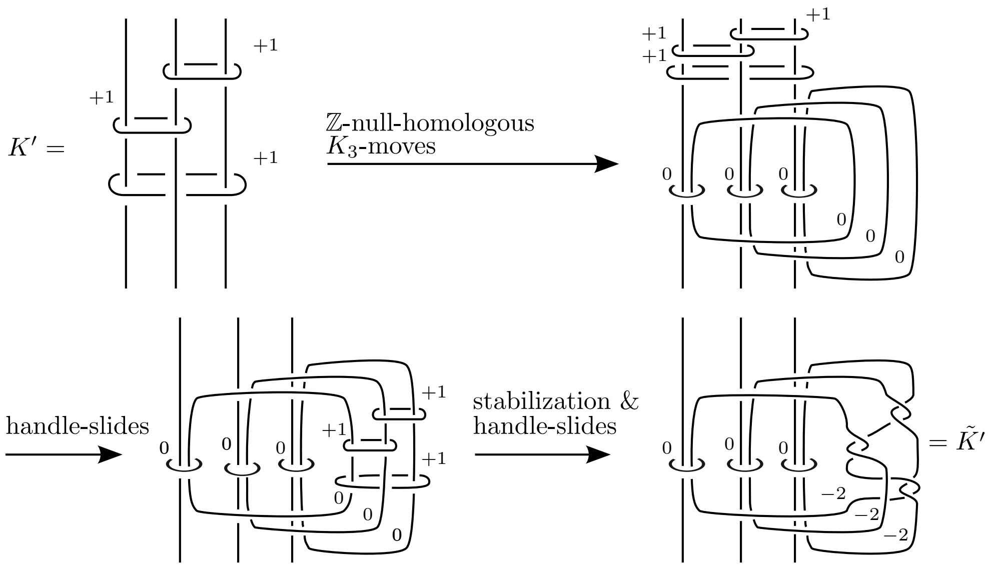



Figure 15: From $K'$ to $\tilde{K}'$.

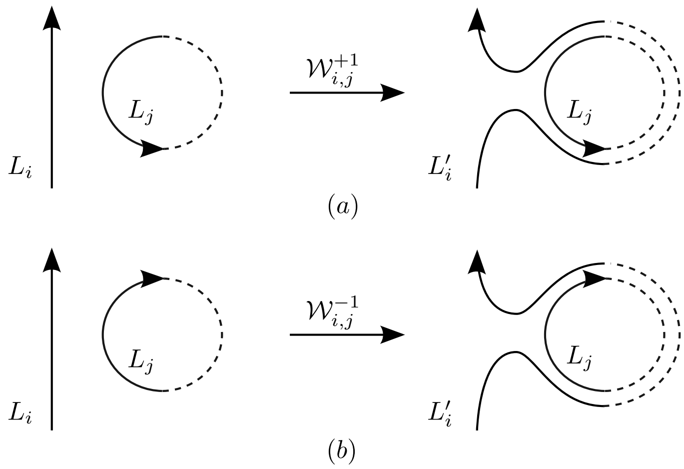



Figure 16: (a) A $\mathcal{W}_{i,j}^{+1}$-move. (b) A $\mathcal{W}_{i,j}^{-1}$-move

- $\mathcal{W}_{i,j}^{\epsilon}$ for $i, j \in \{1, \ldots, n\}$, $i \neq j$, $\epsilon = \pm 1$.

For $e \in \mathcal{E}$, define an $e$-move on $L = L_1 \cup \cdots \cup L_n \in \mathcal{L}_{M,n}$ as follows.

- A $\mathcal{P}_{i,j}$*-move* on $L$ exchanges the order of $L_i$ and $L_j$.
- A $\mathcal{Q}_i$*-move* on $L$ reverses the orientation of $L_i$.
- A $\mathcal{W}_{i,j}^{\epsilon}$*-move* on $L$ is a handle-slide of $L_i$ over $L_j$, see Figure 16.

For $L, L' \in \mathcal{L}_{M,n}$ and $e \in \mathcal{L}_{M,n}$, we mean by $L \xrightarrow{e} L'$ that $L'$ is obtained from $L'$ by an $e$-move. These moves are called the *elementary moves*.

Let $\mathcal{S}_{M,n}$ denote the category such that the objects are the elements of $\mathcal{L}_{M,n}$ and the morphisms from $L \in \mathcal{L}_{M,n}$ to $L' \in \mathcal{L}_{M,n}$ are the sequences of elementary moves

$$S\colon\ L = L_0 \xrightarrow{e_1} L_1 \xrightarrow{e_2} \ldots \xrightarrow{e_p} L_p,$$

$p \geq 0$. The composition of two sequences in $\mathcal{S}_{M,n}$ is given by concatenation of sequences, and the identity $1_L$ of $L \in \mathcal{L}_{M,n}$ is the sequence of length 0.

## 9.4 The functor $\varphi\colon\ \mathcal{S}_{M,n} \to \mathrm{GL}(n; \mathbb{Z})$

There is a functor

$$\varphi\colon\ \mathcal{S}_{M,n} \to \mathrm{GL}(n; \mathbb{Z})$$

from $\mathcal{S}_{M,n}$ to $\mathrm{GL}(n;\mathbb{Z})$, where the group $\mathrm{GL}(n;\mathbb{Z})$ of invertible $n \times n$ matrices with entries in $\mathbb{Z}$ is regarded as a groupoid with one object $*$, such that

$$\begin{aligned}
&\varphi(L \xrightarrow{\mathcal{P}_{i,j}} L') = P_{i,j} := I_n - E_{i,i} - E_{j,j} + E_{i,j} + E_{j,i},\\
&\varphi(L \xrightarrow{\mathcal{Q}_i} L') = Q_i := I_n - 2E_{i,i},\\
&\varphi(L \xrightarrow{\mathcal{W}^{\epsilon}_{i,j}} L') = W^{\epsilon}_{i,j} := I_n + E_{i,j},
\end{aligned}$$

where $E_{i,j} = (\delta_{k,i}\delta_{l,j})_{k,l}$.

**Lemma 9.2** ([11, Lemma 2.2]). *If $L, L' \in \mathcal{L}_{M,n}$ are $\mathbb{Z}$-null-homologous framed links and if $S\colon L \to L'$ is a morphism in $\mathcal{S}_{M,n}$, then we have the following identity for the linking matrices*

$$\mathrm{Lk}(L') = \varphi(S)(\mathrm{Lk}(L'))\varphi(S)^t,$$

*where $(-)^t$ denotes transpose.*

**Theorem 9.3** ([11, Theorem 2.1]). *If a morphism $S\colon L \to L'$ in $\mathcal{S}_{M,n}$ satisfies $\varphi(S) = I_n$, then $L$ and $L'$ are related by a sequence of band-slides.*

Note that a band-slide of an oriented, ordered framed link may be regarded as a morphism in $\mathcal{S}_{M,n}$ of the form $L \xrightarrow{\mathcal{W}^{+1}_{i,j}\mathcal{W}^{-1}_{i,j}} L'$.

## 9.5 Reverse sequences

If

$$S\colon\ L = L^0 \xrightarrow{e_1} L^1 \xrightarrow{e_2} \dots \xrightarrow{e_k} L^k = L'$$

is a sequence in $\mathcal{S}_{M,n}$, then there is the *reverse sequence*

$$\bar{S}\colon\ L' = L^k \xrightarrow{\bar{e}_k} \dots \xrightarrow{\bar{e}_2} L^1 \xrightarrow{\bar{e}_1} L^1 = L,$$

where, for $e \in \mathcal{E}$, $\bar{e} \in \mathcal{E}$ is defined by

$$\bar{\mathcal{P}}_{i,j} = \mathcal{P}_{i,j}, \quad \bar{\mathcal{Q}}_i = \mathcal{Q}_i, \quad (\bar{\mathcal{W}}^{\epsilon}_{i,j}) = \mathcal{W}^{-\epsilon}_{i,j}.$$

We have

$$\varphi(\bar{S}) = \varphi(S)^{-1}.$$

## 9.6 Admissible framed links

An *oriented, ordered admissible framed link* in $M$ of type $(p, q)$, $p, q \geq 0$, is an oriented, ordered, $\mathbb{Z}$-null-homologous framed link

$$L = L_1 \cup \cdots \cup L_p \cup L_{p+1} \cup \cdots \cup L_{p+q} \subset M$$

such that the linking matrix $\mathrm{Lk}(L)$ of $L$ satisfies

$$\mathrm{Lk}(L) = I_{p,q} := I_p \oplus (-I_q),$$

where $I_p$ denotes the identity matrix of size $p$, and $\oplus$ denotes block sum.

For $p, q \geq 0$, let $\mathcal{L}^{\mathrm{adm}}_{M;p,q}$ denote the subset of $\mathcal{L}_{M,p+q}$ consisting of the equivalence classes of oriented, ordered admissible framed links in $M$ of type $(p, q)$. Let $\mathcal{S}^{\mathrm{adm}}_{M;p,q}$ denote the full subcategory of $\mathcal{S}_{M,p+q}$ such that $\mathrm{Ob}(\mathcal{S}^{\mathrm{adm}}_{M;p,q}) = \mathcal{L}^{\mathrm{adm}}_{M;p,q}$.

Let $L, L' \in \mathcal{L}^{\mathrm{adm}}_{M;p,q}$, and suppose that there is a morphism $S\colon L \to L'$ in $\mathcal{S}_{M,p+q}$, i.e., a sequence of elementary moves from $L$ to $L'$. By Lemma 9.2, it follows that

$$I_{p,q} = \varphi(S) I_{p,q} \varphi(S)^t, \tag{9.6.1}$$

hence

$$\varphi(S) \in \mathrm{O}(p, q; \mathbb{Z}) := \{T \in \mathrm{GL}(p + q; \mathbb{Z}) \mid T I_{p,q} T^t = I_{p,q}\}.$$

We use the following result.

**Lemma 9.4** (Wall [19, 1.8]). *If $p, q \geq 2$, then $\mathrm{O}(p, q; \mathbb{Z})$ is generated by the matrices*

$$\begin{aligned}
&P_{i,j} \quad \text{for } i, j \in \{1, \dots, p\},\ i \neq j,\\
&P_{i,j} \quad \text{for } i, j \in \{p + 1, \dots, p + q\},\ i \neq j,\\
&Q_i \quad \text{for } i \in \{1, \dots, p\},\\
&D_{p,q} = \begin{pmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & I_{p-2} & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{q-2} \end{pmatrix}.
\end{aligned} \tag{9.6.2}$$

We consider a sequence of elementary moves on oriented, ordered admissible framed links whose associated matrix is $D_{p,q}$. The matrix

$$D_{2,2} = \begin{pmatrix} -1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix} \in \mathrm{O}(2, 2; \mathbb{Z})$$

is a product of the $W^{\pm 1}_{i,j}$ matrices as

$$D_{2,2} = W^{-1}_{2,1} W^{-1}_{3,1} W_{2,4} W_{3,4} W^{-1}_{4,3} W^{-1}_{1,3} W_{4,2} W_{1,2}$$

Consider the 4-component framed links $l, l', \tilde{l} \in \mathcal{L}_{V_4,4}$ in the handlebody $V_4$ of genus 4 depicted in Figure 17. The handlebody $V_4$ is realized as the complement of the trivial 4-component string link in the cylinder $D^2 \times [0, 1]$.

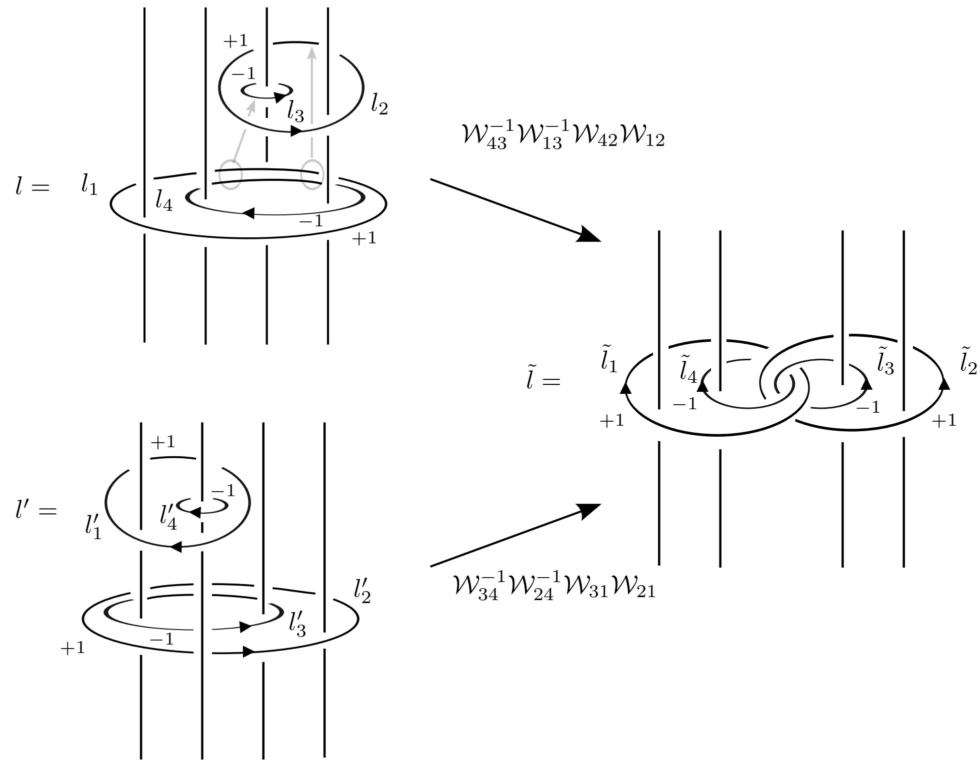



Figure 17:

By applying $\mathcal{W}_{1,2}$, $\mathcal{W}_{4,2}$, $\mathcal{W}_{1,3}^{-1}$, $\mathcal{W}_{4,3}^{-1}$ moves to $l$, we obtain $\tilde{l}$. Similarly, by applying $\mathcal{W}_{2,1}$, $\mathcal{W}_{3,1}$, $\mathcal{W}_{2,4}^{-1}$, $\mathcal{W}_{3,4}^{-1}$ moves to $l'$, we obtain $\tilde{l}$. Thus we have a sequence $\mathcal{D}_{2,2}$ from $l$ to $l'$ such that $\varphi(\mathcal{D}_{2,2}) = D_{2,2}$.

Let $L \in \mathcal{L}^{\mathrm{adm}}_{M;p,q}$ with $p, q \geq 2$. Then we can find an orientation-preserving embedding

$$f\colon V_4 \hookrightarrow M$$

such that $f(l) = L_1 \cup L_2 \cup L_{p+1} \cup L_{p+2}$ as follows.

By adding three edges $c_{1,j}$, $j = 1, 2, 3$, to $l$ in an appropriate way, we obtain a 1-subcomplex $X = l \cup c_{1,2} \cup c_{1,3} \cup c_{1,4}$ of $V_4$, which is a strong deformation retract of $V_4$. Take an embedding $f_X\colon X \hookrightarrow M$ such that $f_X(l) = L_1 \cup L_2 \cup L_{p+1} \cup L_{p+2}$. Then $f_X$ extends to an embedding $f\colon V_4 \hookrightarrow M$ with the desired property.

Set $L' = L'_1 \cup \cdots \cup L'_{p+q}$, where

$$L'_1 = f(l'_1), \quad L'_2 = f(l'_2), \quad L'_{p+1} = f(l'_3), \quad L'_{p+2} = f(l'_4),$$
$$L'_i = L_i \quad \text{for } i \in \{3, \ldots, p, p+3, \ldots, p+q\}.$$

Then there is a sequence $\mathcal{D}\colon L \to L'$, corresponding to the sequence $\mathcal{D}_{2,2}$, such that $\varphi(\mathcal{D}) = D_{p,q}$. Similarly, given $L \in \mathcal{L}^{\mathrm{adm}}_{M;p,q}$, there is a sequence $\mathcal{D}^{-1}\colon L \to L'$ such that $\varphi(\mathcal{D}^{-1}) = D^{-1}_{p,q}$. In these situations, $L$ and $L'$ are said to be related by a $\mathcal{D}^{\pm 1}$*-move.*

Now, we can prove the following.

**Proposition 9.5.** *Let $L, L' \in \mathcal{L}^{\mathrm{adm}}_{M;p,q}$ with $p, q \geq 2$. Suppose that there is a morphism $S\colon L \to L'$ in $\mathcal{S}_{M,p+q}$, i.e., a sequence of elementary moves from $L$ to $L'$. Then $L$ and $L'$ are related by a sequence of*

- *band-slides,*
- *$\mathcal{P}_{i,j}$-moves for $i, j \in \{1, \dots, p\}, i \neq j$ and for $i, j \in \{p+1, \dots, p+q\}, i \neq j$,*
- *$\mathcal{Q}_i$-moves for $i \in \{1, \dots, p\}$,*
- *$\mathcal{D}^{\pm 1}$-moves.*

*Proof.* Express $\varphi(S) \in \mathrm{O}(p, q; \mathbb{Z})$ as

$$\varphi(S) = x_k \dots x_2 x_1, \quad k \geq 0,$$

where each $x_i$ is one of the generators given in (9.6.2) or its inverse. We can construct a sequence

$$T\colon\ L = L_0 \xrightarrow{x_1} L_1 \xrightarrow{x_2} \dots \xrightarrow{x_k} L_k = L''$$

such that $L_0, \dots, L_k \in \mathcal{L}^{\mathrm{adm}}_{M;p,q}$, and for each $m = 1, \dots, k$, $L_m$ is obtained from $L_{m-1}$ by either a $\mathcal{P}_{i,j}$-move, a $\mathcal{Q}_i$-move or a $\mathcal{D}^{\pm 1}$-move corresponding to $x_m$. We may regard $T$ as a sequence from $L$ to $L''$ of elementary moves, i.e., a morphism from $L$ to $L''$ in $\mathcal{S}_{M,p+q}$, by replacing each $\mathcal{D}^{\pm 1}$ move in $T$ with the corresponding sequence of 8 $\mathcal{W}^{\pm 1}_{i,j}$-moves. Thus, $L$ and $L''$ are related by a sequence of moves listed in the proposition (without band-slides).

Now the composite sequence $T\bar{S}\colon L' \to L''$ satisfies $\varphi(T\bar{S}) = \varphi(T)\varphi(S)^{-1} = I_{p+q}$. Hence, it follows from Theorem 9.3 that there is a sequence of band-slides from $L'$ to $L''$.

Hence it follows that there is a sequence from $L$ to $L'$ of moves listed in the proposition. □

## 9.7 Proof of Proposition 9.1

We have to prove that (ii) implies (i) in Proposition 9.1.

Throughout this section, $M$ is a compact, connected, oriented 3–manifold with non-empty boundary such that $H_1 M \cong \mathbb{Z}^r$ is free abelian. Let $L$ and $L'$ be two admissible framed links in $M$. Let

$$S\colon\ L = L^0 \to L^1 \to \cdots \to L^k = L' \tag{9.7.1}$$

be a sequence of $\mathbb{Z}$-null-homologous framed links between $L$ and $L'$ such that, for each $i = 1, \dots, k$, $L^i$ is obtained from $L^{i-1}$ by either stabilization, handle-slide, $\mathbb{Z}$-null-homologous $K_3$-move or IHX-move.

### 9.7.1 Eliminating IHX-moves

Let $h_S\colon M_L \xrightarrow{\cong} M_{L'}$ be the diffeomorphism associated to the sequence $S$. Set

$$\eta(S) := \theta_4(\mathrm{tr}((W_{L'}^M)^{-1} \circ C_{h_S} \circ W_L^M)) \in H_4(H_1 M).$$

Note that $H_4(H_1M)$ is finitely generated by elements of the form $p(x_1 \wedge \cdots \wedge x_4)$ with $x_1, \ldots, x_4 \in H_1M$. Hence, using Proposition 8.2, we can construct a sequence $T$ of admissible IHX-moves

$$T\colon\ L' = K^0 \to K^1 \to \cdots \to K^m = L''$$

from $L'$ to an admissible framed link $L''$ such that

- $\eta(T) = -\eta(S)$.
- there are orientation-preserving embeddings $f_1, \ldots, f_m\colon\ V_4 \hookrightarrow M \backslash L'$ with mutually disjoint images such that

  $$K^i = L' \cup f_1(L_{IHX}^{\mathrm{adm}}) \cup \cdots \cup f_i(L_{IHX}^{\mathrm{adm}})$$

  for $i = 0, \ldots, m$.

Then

$$\begin{aligned}
\eta(TS) &= \theta_4(\mathrm{tr}((W_{L''}^M)^{-1} \circ C_{h_{TS}} \circ W_L^M)) \\
&= \theta_4(\mathrm{tr}((W_{L''}^M)^{-1} \circ C_{h_T} \circ C_{h_S} \circ W_L^M)) \\
&= \theta_4(\mathrm{tr}((W_{L''}^M)^{-1} \circ C_{h_T} \circ (W_{L'}^M) \circ (W_{L'}^M)^{-1} \circ C_{h_S} \circ W_L^M)) \\
&= \theta_4(\mathrm{tr}((W_{L''}^M)^{-1} \circ C_{h_T} \circ (W_{L'}^M))) + \theta_4(\mathrm{tr}(((W_{L'}^M)^{-1} \circ C_{h_S} \circ W_L^M)) \\
&= \eta(T) + \eta(S) = 0,
\end{aligned}$$

where $h_{TS}\colon\ M_L \to M_{L''}$ is the diffeomorphism associated to the composite sequence $TS\colon\ L \to L''$. Then, by Theorem 3.3 with $N = [\pi_1 M, \pi_1 M]$, it follows that $L$ and $L''$ are related by a sequence of stabilization, handle-slides, and $[\pi_1 M, \pi_1 M]$-moves, i.e., $\mathbb{Z}$-null-homologous $K_3$-moves.

Thus, we may assume without loss of generality that there are no IHX-moves in the sequence $S$.

### 9.7.2 Eliminating $\mathbb{Z}$-null-homologous $K_3$-moves

We have a sequence $S$ in (9.7.1) of stabilization, handle-slides, $\mathbb{Z}$-null-homologous $K_3$-moves and ambient isotopies. Unlike the other part of the paper, here, we distinguish two ambient isotopic framed links. If the $i$th move $L^{i-1} \to L^i$ is either stabilization, handle-slide, or $\mathbb{Z}$-null-homologous $K_3$-move, then we specify a handlebody $V^i \subset M$ in which the move takes place. By modifying the sequence $S$ if necessary, we can choose such $V^i$ sufficiently thin so that the union $\left(\bigcup_{i=0}^{k} L^k\right) \cup \bigcup_i V^i$ is contained in a handlebody $V \subset \operatorname{int} M$. Note that the homomorphism

$$[\pi_1(M \setminus V), \pi_1(M \setminus V)] \to [\pi_1 M, \pi_1 M]$$

induced by inclusion $M \setminus V \subset M$ is surjective.

Since $\pi_1 M$ is finitely generated, the commutator subgroup $[\pi_1 M, \pi_1 M]$ is generated by the conjugates in $\pi_1 M$ of finitely many elements $x_1, \ldots, x_t \in [\pi_1 M, \pi_1 M]$, $t \geq 0$. Let $\tilde{x}_t \in [\pi_1(M \setminus V), \pi_1(M \setminus V)]$ be a lift of $x_t$. We can find an admissible framed link

$$K = K_1^+ \cup K_1^- \cup \cdots \cup K_t^+ \cup K_t^-$$

in $M \setminus V$ satisfying the following conditions.

(1) The (free) homotopy classes of $K_i^+$ and $K_i^-$ are $\tilde{x}_i$.

(2) There are $t$ disjoint annuli $A_1, \ldots, A_t$ in $M \setminus V$ such that $\partial A_j = K_j^+ \cup K_j^-$,

(3) The framing of $K_i^{\pm}$ is $\pm 1$.

Set $\tilde{L} = L \cup K$, $\tilde{L}' = L' \cup K$ and $\tilde{L}^i = L^i \cup K$, $i = 0, \ldots, k$. Then $\tilde{L}$ (resp. $\tilde{L}'$) is obtained from $L$ (resp. $L'$) by $k$ pair-moves. Thus, it suffices to show that for each $i = 1, \ldots, k$, $\tilde{L}^{i-1}$ and $\tilde{L}^i$ are related by sequence of stabilizations, handle-slides and ambient isotopies. Thus, we may safely assume that $t = 1$.

If $L$ and $L'$ are related by either a stabilization or a handle-slide in $M \setminus (A_1 \cup \cdots \cup A_t)$, then clearly $\tilde{L}$ and $\tilde{L}'$ are related by a stabilization or a handle-slide.

If $L$ and $L'$ are related by a $\mathbb{Z}$-null-homologous $K_3$-move, then let us assume that $L' = L \cup J \cup J'$ is obtained from $L'$ by adding a $\mathbb{Z}$-null-homologous component $J$ and a small 0-framed meridian $J'$ of $J$. (Of course, the case of $\mathbb{Z}$-null-homologous $K_3$-move in the other direction is similar.) Since the homotopy classes of the $K_i^{\pm}$ generate $[\pi_1 M, \pi_1 M]$ normally in $\pi_1 M$, it follows that we can slide $J$ over the $K_i^{\pm}$ several times to make $J$ null-homotopic in $M$. Then there is a sequence from $J$ and an unknot of crossing changes of $J$ with any components of the framed link other than $J'$. Such crossing changes can be realized by handle-slides of link components over $J'$. Thus we may assume that $J \cup J'$ is a Hopf link such that $J'$ is of framing 0. It is well known that $J \cup J'$ is related to the empty link by a sequence of stabilizations and handle-slides. Hence, it follows that $L$ and $L'$ are related by a sequence of stabilizations and handle-slides.

If $L$ and $L'$ are ambient isotopic in $M$, then they are related by a sequence of

- ambient isotopies in $M \setminus (A_1 \cup \cdots \cup A_t)$,
- crossing changes of a component with some $A_j$.

We may assume without loss of generality that $L$ and $L'$ are related by one of these moves. If $L$ and $L'$ are ambient isotopic in $M \setminus (A_1 \cup \cdots \cup A_t)$, then $\tilde{L}$ and $\tilde{L}'$ are ambient isotopic in $M$. If $L$ and $L'$ are related by a crossing changes of a component $L_c$ of $L$ with $A_i$, then $\tilde{L}$ and $\tilde{L}'$ are related by two handle-slides. (Here, we first slide $L_c$ over $K_i^+$, and the we slide it over $K_i^-$.)

### 9.7.3 Eliminating stabilizations

Now, $L$ and $L'$ are related by a sequence $S$ in (9.7.1) of stabilizations and handle-slides.

It is well known that we can exchange the order of consecutive stabilizations and handle-slides to obtain a new sequence

$$S'\colon\ L \to \cdots \to \tilde{L} \to \cdots \to \tilde{L}' \to \cdots \to L',$$

where $\tilde{L}$ is obtained from $L$ by adding isolated $\pm 1$-framed unknots by stabilizations, and $\tilde{L}'$ is obtained from $\tilde{L}$ by a sequence of handle-slides, and $L'$ is obtained from $\tilde{L}'$ by removing isolated $\pm 1$-framed unknots by stabilizations. Note that $\tilde{L}$ and $\tilde{L}'$ are admissible.

We may assume that $\tilde{L}$ (and hence $\tilde{L}'$) is admissible of type $(p,q)$ with $p,q \geq 2$, since if not we can add the number of components by using stabilizations.

Thus, we have only to consider the sequence $\tilde{L} \to \cdots \to \tilde{L}'$ of handle-slides, where $\tilde{L}$ and $\tilde{L}'$ are admissible of type $(p,q)$ with $p,q \geq 2$.

### 9.7.4 Reduction to $\mathcal{D}^{\pm 1}$-moves

Suppose that $L$ and $L'$ are related by a sequence of handle-slides and that $L$ and $L'$ are admissible of type $(p,q)$ with $p,q \geq 2$.

We fix an orientation and ordering of $L$ and $L'$. Then $L$ and $L'$ are, as oriented, ordered framed links, related by elementary moves as defined in Section 9.3. Then by Proposition 9.5 it follows that $L$ and $L'$ are related by a sequence of moves listed in Proposition 9.5. Hence $L$ and $L'$, as non-ordered, non-oriented framed links, are related by a sequence of band-slides and $\mathcal{D}^{\pm 1}$-moves, where each $\mathcal{D}^{\pm 1}$-move can be applied to any 4-component sublinks of framings $+1, +1, -1, -1$ by assuming any orientation.

### 9.7.5 $\mathcal{D}^{\pm 1}$-moves and lantern-moves

Now it suffices to prove the following lemma.

**Lemma 9.6.** *Suppose that admissible framed links $L$ and $L'$ are related by a $\mathcal{D}^{\pm 1}$-move. Then there is a sequence between $L$ and $L'$ of two lantern-moves, and several pair-moves.*

*Proof.* Suppose that $L'$ is obtained from $L$ by a $\mathcal{D}^{+1}$-move.

Let $l = l_1 \cup l_2 \cup l_3 \cup l_4 \subset L$ be the sublink of $L$ involved in the $\mathcal{D}^{+1}$-move. In Figure 18(a), the framed link $l$ in a genus 4 handlebody $V_4$ in $M$ is depicted. Here, as usual, $V_4$ is identified with the complement of a trivial string link $\gamma$ in the cylinder $D^2 \times [0,1]$. Recall that the meridian to each strand of $\gamma$ is null-homologous in $M$, and have zero linking numbers with each component of $L$.

Then $L'$ is obtained from $L$ by removing $l$ and adding a 4-component sublink $l' = l'_1 \cup l'_2 \cup l'_3 \cup l'_4$ in Figure 18(e). We can go from (a) to (e) by using pair-moves and lantern-moves as follows.

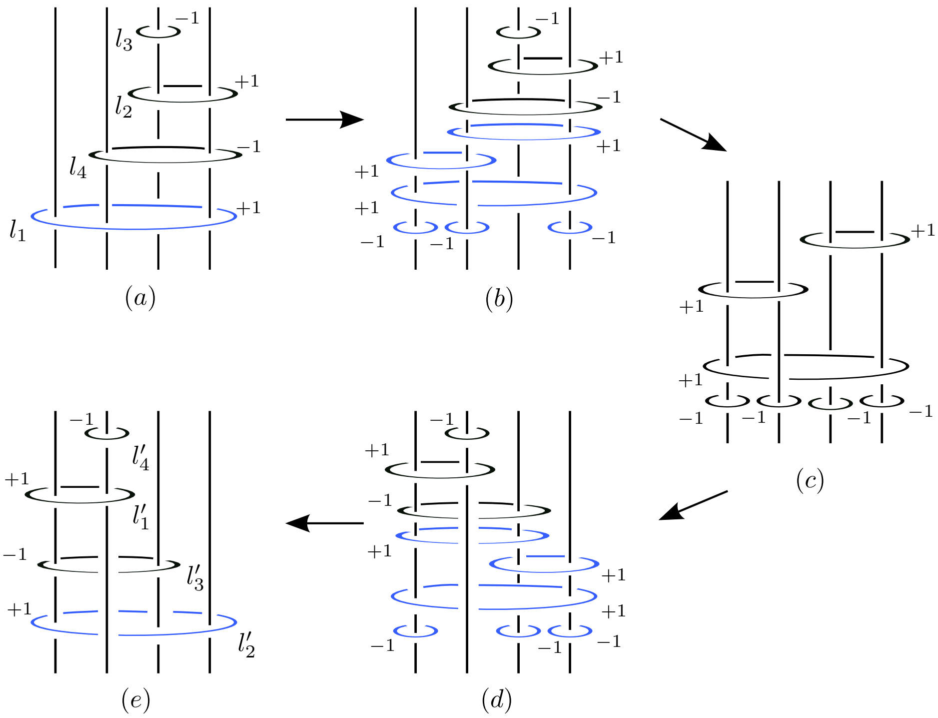



Figure 18:

Starting at (a), we first apply pair-moves along the meridians to the first, second and fourth strands, and then apply a lantern-move involving $l_1$ to obtain (b). Next, we arrive at (c) by applying a pair-move to remove two components which links with the second and fourth strands.

Similarly, we can go from (e) to (c). We get from (e) to (d) by using pair-moves along the meridians to the first, third and fourth strands, and a lantern-move involving $l'_2$. Then we get at (e) by one pair-move. $\square$

*Remark* 9.7. A lantern-move can be realized by stabilizations, pair-moves, and one $\mathcal{D}^{\pm 1}$-move. To see this, one embeds $V_4$ in $M$ in such a way that the meridian to the third strand of $\gamma$ is mapped to a 0-framed unknot bounding a disk which does not intersect the other components of the framed link. This amounts to removing the third strand of $\gamma$ in the definition of $\mathcal{D}^{\pm 1}$-move. Then this special $\mathcal{D}^{\pm 1}$-move is realized by one lantern-move up to stabilizations and pair-moves.

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# Appendix A

# The map $\rho$

This appendix is devoted to an extra detail related to Chapter 5. In the case of 3–manifolds with boundaries we showed in Chapter 5 Lemma 2.1 that the map $\rho\colon W \to K(\pi_1(W_L), 1)$ exists, if the fundamental groupoid diagrams commute. We now show that the existence of the map $\rho$ is given **if and only if** the fundamental groupoid diagrams commute.

We use the notation given in Chapter 5 Section 2. Consider two framed links $L$ and $L'$ in a compact, connected, oriented 3–manifold $M$ with boundary $\partial M = F_1 \sqcup \cdots \sqcup F_n$. Pick basepoints $p_k \in F_k$ for $k = 1, \dots, n$. Assume there is a homeomorphism $h\colon M_L \to M_{L'}$ relative to the boundary.

**Proposition 26.** *The following are equivalent:*

*(i) There exist isomorphisms $f_k\colon \pi_1(W_L; p_1, p_k) \to \pi_1(W_{L'}; p_1, p_k)$ such that the following diagram commutes for $k = 1, \dots, n$.*

$$\begin{tikzcd}
\pi_1(M_L; p_1^L, p_k^L) \arrow[rr, "h_k"] \arrow[d, "i'_k"'] & & \pi_1(M_{L'}; p_1^{L'}, p_k^{L'}) \arrow[d, "i'_k"] \\
\pi_1(W_L; p_1, p_k) \arrow[rr, "f_k"] & & \pi_1(W_{L'}; p_1, p_k) \\
& \pi_1(M; p_1, p_k) \arrow[ul, "i_k"] \arrow[ur, "i_k"'] &
\end{tikzcd} \tag{A.1}$$

*(ii) The following diagram commutes.*

$$\begin{tikzcd}
\pi_1(\partial W_L) \arrow[r, "u'_*"] \arrow[d, "u_*"'] & \pi_1(W_{L'}) \arrow[d, "j'_*"] \\
\pi_1(W_L) \arrow[r, "j_*"] \arrow[ur, "f_1"] & \pi_1(W)
\end{tikzcd} \tag{A.2}$$

*(iii) There exists a map $\rho : W \to K(\pi, 1)$ such that the following diagram commutes.*

$$\begin{tikzcd}
\partial W_L \arrow[r, "u'"] \arrow[d, "u"'] & W_{L'} \arrow[d, "j'"'] \arrow[ddr, "\tilde{\rho}_{L'}"] & \\
W_L \arrow[r, "j"] \arrow[drr, "\rho_L"'] & W \arrow[dr, "\rho"] & \\
& & K(\pi_1(W_L), 1)
\end{tikzcd} \tag{A.3}$$

*Proof.* The cases (i) implies (ii) implies (iii) are shown in Chapter 5 Lemma 2.1 and the explanations right before.

We now show (iii) implies (ii). Assume, there is a map $\rho : W \to K(\pi, 1)$ such that Diagram (A.3) commutes. For the induced maps on the homotopy groups we get

$$\rho_* j_* = id_{\pi_1(W_L)}, \tag{A.4}$$
$$\rho_* j'_* = (f_1)^{-1}, \tag{A.5}$$
$$\rho_* j'_* f_1 = \rho_* j_*, \tag{A.6}$$

where Equation (A.6) is obtained from (A.4) and (A.5). From (A.4) and (A.5) we see that $j_*$ and $j'_*$ are injective. The square is a push-out diagram and the maps $u_*$ and $u'_*$ are surjective therefore $j_*, j'_*$ are surjective. Hence, $j_*, j'_*$ are isomorphisms. It follows that $\rho_*$ is an isomorphism and we obtain $j'_* f_1 = j_*$ from (A.6). Since the square commutes and $j_*, j'_*$ are isomorphisms we get $f_1 u_* = u'_*$.

It remains to proof that (ii) implies (i). Commutativity of Diagram (A.2) implies commutativity of Diagram (A.1) for $k = 1$. For $k > 1$, we need the following construction. Let $\hat{\beta}_k$ be a path in $M_L$ from $p_1^L$ to $p_k^L$ for each $k = 2, \ldots, n$. Define maps

$$\beta_k : \pi_1(M_L; p_1^L) \to \pi_1(M_L; p_1^L, p_k^L), [g] \mapsto [g \circ \hat{\beta}_k] \tag{A.7}$$

where the composition is given by the composition of paths in $M_L$. Consider the $\hat{\beta}_k$ as paths in $W_L$ and $\hat{\beta}'_k := h(\hat{\beta}_k)$ as paths in $M_{L'}$. Define maps from $\pi_1(W_L)$ and $\pi_1(M_{L'})$ to the fundamental groupoids analogous to the maps $\beta_k$.

We define maps $f_k := \beta'_k f \beta_k^{-1}$ to obtain the following commutative diagram for $k = 2, \ldots, n$.

$$\begin{tikzcd}
\pi_1(M_L; p_1^L, p_k^L) \arrow[rrr, "h_*"] \arrow[ddd, "i'_*"'] & & & \pi_1(M_{L'}; p_1^{L'}, p_k^{L'}) \arrow[ddd, "i'_*"] \\
& \pi_1(M_L; p_1^L) \arrow[ul, "\beta_k"'] \arrow[r, "h_*"] \arrow[d, "i'_*"'] & \pi_1(M_{L'}; p_1^{L'}) \arrow[ur, "\beta'_k"] \arrow[d, "i'_*"] & \\
& \pi_1(W_L; p_1) \arrow[r, "f_1"] \arrow[dl, "\beta_k"'] & \pi_1(W_{L'}; p_1) \arrow[dr, "\beta'_k"] & \\
\pi_1(W_L; p_1, p_k) \arrow[rrr, "f_k"] & & & \pi_1(W_{L'}; p_1, p_k)
\end{tikzcd} \tag{A.8}$$

To show that the lower triangle of Diagram (A.1) commutes we consider the following commutative double triangles for $k = 2, \ldots, n$. Here, the innermost triangle commutes by our assumption and the other polygons commute by definition.

$$\begin{tikzcd}
\pi_1(W_L; p_1, p_k) \arrow[rrrr, "f_k"] & & & & \pi_1(W_{L'}; p_1, p_k) \\
& \pi_1(W_L; p_1) \arrow[ul, "\beta_k"'] \arrow[rr, "f_1"] & & \pi_1(W_{L'}; p_1) \arrow[ur, "\beta'_k"] & \\
& & \pi_1(\partial W_L; p_1) \arrow[ul, "u_*"] \arrow[ur, "u'_*"'] \arrow[d, "\beta_k"'] & & \\
& & \pi_1(\partial W_L; p_1, p_k) \arrow[uuull, "u_k"] \arrow[uuurr, "u'_k"'] & &
\end{tikzcd}$$

The inclusion $M \subset \partial W_L$ then induces the commutativity of the lower triangle of Diagram (A.1) for $k = 2, \ldots, n$. $\square$

**Corollary 27.** *If Diagram* (A.2) *commutes the maps $j_*$ and $j'_*$ are isomorphisms.*

*Proof.* Consider $v \in \pi_1(W_L)$. Since $u_*$ is surjective there exists an $x \in \pi_1(\partial W_L)$ with $u_*(x) = v$. Because the square in (A.2) commutes we have $j_*(u_*(x)) = j'_*(u'_*(x))$. Thus, $j_*(v)$ is in the image of $j'_*$. Since Diagram (A.2) is induced by a pushout diagram $j'_*$ is surjective. Surjectivity of $j_*$ is proven analogously.
Now, let $v \in \pi_1(W_L)$ with $j(v) = 1 \in \pi_1(W)$. By the surjectivity of $u_*$ there exists an $x \in \pi_1(\partial W_L)$ with $u_*(x) = v$. Then $j_*(u_*(x) = 1 = j'_*(u'_*(x))$. We can can assume $u'_*(x) = 1 \in \pi_1(W_{L'})$. Since $f_1 u_* = u'_*$ we get $f_1(u_*(x)) = u'_*(x) = 1$ and because $f_1$ is an isomorphism it follows that $u_*(x) = v = 1$. Therefore $j_*$ is injective and analogously $j'_*$. $\square$

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